LICHNEROWICZ-JACOBI COHOMOLOGY AND HOMOLOGY OF JACOBI MANIFOLDS: MODULAR CLASS AND DUALITY

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Abstract

Lichnerowicz-Jacobi cohomology and homology of Jacobi manifolds are reviewed. We present both in a unified approach using the representation of the Lie algebra of functions on itself by means of the hamiltonian vector fields. The use of the associated Lie algebroid allows to prove that the Lichnerowicz-Jacobi cohomology and homology are invariant under conformal changes of the Jacobi structure and to stablish the duality between Lichnerowicz-Jacobi cohomology and homology when the modular class vanishes. We also compute the Lichnerowicz-Jacobi cohomology and homology for a large variety of examples.

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Contents

1 Introduction

Since their introduction by Lichnerowicz in [38, 39], Poisson and Jacobi manifolds have deserved a lot of interest in the mathematical physics literature. Indeed, the need to use more general phase spaces for hamiltonian systems lead to the consideration of Poisson brackets of non-constant rank, and, more than this, brackets which do not satisfy Leibniz rule (Jacobi brackets).

From the viewpoint of Differential Geometry, both structures are of great interest. The local and global structures of Poisson and Jacobi manifolds were ellucidated by Dazord, Guedira, Lichnerowicz, Marle, Weinstein and many others ([12, 21, 54]; see also [3, 36, 52]). A Poisson manifold is basically made of symplectic pieces, but the structure of a Jacobi manifold is more complicated, and it is made of pieces which are contact or locally conformal symplectic manifolds.

The Poisson structure of a Poisson manifold M allows to define some homology and cohomology operators. Indeed, the Poisson bivector of M determines the so-called Lichnerowicz-Poisson cohomology (LP-cohomology) and the 1-differentiable Chevalley-Eilenberg cohomology, which can be alternatively described as the cohomologies of two subcomplexes of the Chevalley-Eilenberg complex associated with the Lie algebra of differentiable functions endowed with its Poisson bracket (see [38]). The first of these subcomplexes consists of the linear skew-symmetric multidifferential operators which are derivations in each argument with respect to the usual product of functions, that is, the multivectors on M. The second one consists of the 1-differentiable cochains, that is, the linear skew-symmetric multidifferential operators of order 1 in each argument. Note that the space of k-cochains of this subcomplex is isomorphic to $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$, where $\mathcal{V}^r(M)$ is the space of r-vectors on M. Computation of Poisson cohomology is generally quite difficult. For regular Poisson manifolds and for the Lie-Poisson structure on the dual space of the Lie algebra of a compact Lie group, some results were obtained in [18, 19, 51, 58]. On the other hand, we remark that the k-th LP-cohomology group has interesting interpretations for the first few values of k. Moreover, these cohomology groups allow to describe important results about the geometric quantization and the deformation quantization of Poisson manifolds (for more information, we refer to [52] and to the recent survey [56]; see also the references therein). The Poisson tensor of M also allows to define the canonical homology operator on forms (see [5, 29]). The duality between the canonical homology and the LP-cohomology is directly related to the vanishing of the modular class introduced by Weinstein [55] (see also [6, 14, 59]).

The situation for a Jacobi manifold M is more involved. Note that the Jacobi bracket of functions on M is not a derivation in each argument with respect to the usual product of functions (this is the difference with the Poisson bracket). It is only a linear skew-symmetric 2-differential operator of order 1 or, in other words, a 1-differentiable 2-cochain in the Chevalley-Eilenberg complex of the Lie algebra of functions. Thus, for the manifold M, we have two possibilities. The first one is to consider the representation of the Lie algebra of functions on itself given by the Jacobi bracket. The resultant cohomol-

ogy, the Chevalley-Eilenberg cohomology of the Lie algebra of functions, was studied by Guedira and Lichnerowicz [21, 39]. Particularly, Guedira and Lichnerowicz studied the 1-differentiable Chevalley-Eilenberg cohomology, that is, the cohomology of the subcomplex of the Chevalley-Eilenberg complex which consists of the 1-differentiable cochains. The second possibility is to consider the representation of the Lie algebra of functions on itself given by the action of the hamiltonian vector fields. The resultant cohomology was termed by the authors, in [33, 34], the H-Chevalley-Eilenberg cohomology of M. As in the case of the Chevalley-Eilenberg complex of M, one can consider also the subcomplex of the 1-differentiable cochains. The cohomology of this subcomplex was termed the Lichnerowicz-Jacobi cohomology (LJ-cohomology, for brevity) of M (see [33, 34]). If M is a Poisson manifold then the Chevalley-Eilenberg cohomology and the H-Chevalley-Eilenberg cohomology coincide and the 1-differentiable Chevalley-Eilenberg cohomology is just the LJ-cohomology. On the other hand, the Lichnerowicz-Poisson complex of M is isomorphic to a subcomplex of the Lichnerowicz-Jacobi complex. The H-Chevalley-Eilenberg cohomology and the LJ-cohomology of a Jacobi manifold M play an important role in the geometric quantization of M and in the study of the existence of prequantization representations for complex line bundles over M (for more details, see [33, 34]).

The LJ-cohomology can be also described using the Lie algebroid associated with the Jacobi manifold. Indeed, it is just the Lie algebroid cohomology with trivial coefficients (see [33, 34, 53]). Using this fact we prove, in this paper, that the LJ-cohomology is invariant under conformal changes of the Jacobi structure. Moreover, we show that in many cases, the LJ-cohomology can be related with the de Rham cohomology of the manifold, and in some cases it results a topological invariant.

On the other hand, using again the representation of the Lie algebra of functions on itself given by the hamiltonian vector fields, we introduce the H-Chevalley-Eilenberg homology of a Jacobi manifold M. The H-Chevalley-Eilenberg homology operator permits to define an homology operator δ on the complex $\Omega^*(M)\oplus\Omega^{*-1}(M)$, where $\Omega^k(M)$ is the space of k-forms on M, $\Omega^*(M) = \bigoplus_{k=1}^{n} \Omega^k(M)$ and n is the dimension of M. The resultant homology is termed the Lichnerowicz-Jacobi homology (LJ-homology). If M is a Poisson manifold the canonical homology of M is isomorphic to the homology of a subcomplex of the Lichnerowicz-Jacobi complex.

It is easy to prove that the LJ-homology of a Jacobi manifold M is isomorphic to the Jacobi homology introduced by Vaisman [53], which is described using the Lie algebroid associated with M. In fact, the Jacobi homology is the Lie algebroid homology with respect to a flat connection on the top exterior power of the jet bundle $J^1(M,\mathbb{R})$. Using this result, we show, in this paper, that the LJ-homology is invariant under conformal changes of the structure. In [53], Vaisman also introduces the modular class of M as an element of the first LJ-cohomology group. Moreover, he proves that if such a class is null (that is, M is unimodular) then the Lie algebroid cohomology with trivial coefficients and the Jacobi homology are dual one each other. However, as we will show in this paper, there exist

important examples of Jacobi manifolds which are not unimodular. In these cases we will describe the LJ-homology and we will conclude that in most of them, the LJ-cohomology and the LJ-homology are not, in general, dual one each other.

The paper is organized as follows.

Section 2 is introductory, and it contains some generalities about Jacobi manifolds: definitions, examples and the construction of the Lie algebroid canonically associated to any Jacobi manifold. In Section 3, we first recall the notions of H-Chevalley-Eilenberg cohomology and LJ-cohomology of a Jacobi manifold (M, Λ, E). In particular, we recall that the LJ-cohomology can be described as the cohomology of the Lie algebroid associated with M (see [33, 34, 53]) or alternatively introducing the operator $\sigma : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow \mathcal{V}^{k+1}(M) \oplus \mathcal{V}^k(M)$, given by $\sigma(P, Q) = (-[\Lambda, P] + kE \wedge P + \Lambda \wedge Q, [\Lambda, Q] - (k-1)E \wedge Q + [E, P])$, where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket (see [33, 34]). The first description allows us to prove that the LJ-cohomology is invariant under conformal changes of the Jacobi structure. The rest of the section is devoted to study the LJ-cohomology for different examples of Poisson structures (symplectic and Lie-Poisson structures and a quadratic Poisson structure on $\mathbb{R}^2$) and of Jacobi structures. In particular, the LJ-cohomology of contact and locally conformal symplectic manifolds is extensively studied. We also consider another interesting example: the Jacobi structure of the unit sphere of a real Lie algebra of finite dimension. In Table I, we summarize the main results obtained about the LJ-cohomology of the different examples of Jacobi manifolds.

In Section 4, we introduce and study the LJ-homology of a Jacobi manifold (M, Λ, E). First of all, the H-Chevalley-Eilenberg homology of M is defined as the homology of the Lie algebra of functions on M with respect to the representation given by the hamiltonian vector fields. Since every k-chain of the H-Chevalley-Eilenberg complex defines a pair (α, β), with α a k-form and β a $(k-1)$-form, the H-Chevalley-Eilenberg boundary operator induces a boundary operator $\delta : \Omega^k(M) \oplus \Omega^{k-1}(M) \longrightarrow \Omega^{k-1}(M) \oplus \Omega^{k-2}(M)$ given by $\delta(\alpha, \beta) = (i(\Lambda)d\alpha - di(\Lambda)\alpha + ki_E\alpha + (-1)^k \mathcal{L}_E\beta, i(\Lambda)d\beta - di(\Lambda)\beta + (k-1)i_E\beta + (-1)^k i(\Lambda)\alpha)$, where $i(\Lambda)$ denotes the contraction by Λ and $\mathcal{L}$ is the Lie derivative operator. The resultant homology is called Lichnerowicz-Jacobi homology (LJ-homology). As we have indicated above, there is an alternative description of the LJ-homology which is due to Vaisman [53]. This description allows us to prove an important first result: the invariance of the LJ-homology under conformal changes of the Jacobi structure. The rest of the section is devoted to study the LJ-homology of the different examples of Jacobi manifolds considered in Section 3. We show that the symplectic manifolds, the dual space of a unimodular real Lie algebra $\mathfrak{g}$ (endowed with the Lie-Poisson structure) and the unit sphere on $\mathfrak{g}$ are unimodular Jacobi manifolds. Thus, using the results of Section 3 and the results of Vaisman [53] on the duality between the LJ-cohomology and the LJ-homology, we describe the LJ-homology of the above examples of Jacobi manifolds (at least for the case when $\mathfrak{g}$ is the Lie algebra of a compact Lie group). On the other hand, the contact manifolds and the locally (non-globally) conformal symplectic manifolds are not unimodular Jacobi manifolds. In fact, we deduce that in these cases the LJ-homology and the LJ-cohomology are not, in general, dual one

each other. In Table II, we summarize the main results obtained on the LJ-homology (and its relation with the LJ-cohomology) of the different examples of Jacobi manifolds.

2 Jacobi manifolds

All the manifolds considered in this paper are assumed to be connected. Moreover, if M is a differentiable manifold, we will use the following notation:

- $C^\infty(M,\mathbb{R})$ is the algebra of C^∞ real-valued functions on M.
- $\mathfrak{X}(M)$ is the Lie algebra of the vector fields on M.
- $\Omega^k(M)$ is the space of k-forms on M.
- $\mathcal{V}^k(M)$ is the space of k-vectors on M.

2.1 Local Lie algebras and Jacobi manifolds

A *Jacobi structure* on a n-dimensional manifold M is a pair (Λ, E) where Λ is a 2-vector and E a vector field on M satisfying the following properties:

$$[\Lambda,\Lambda] = 2E\wedge\Lambda\ , \qquad \mathcal{L}_E\Lambda = [E,\Lambda] = 0\ . \tag{2.1}$$

Here $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket ([3, 52]) and $\mathcal{L}$ is the Lie derivative operator. The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f,g\} = \Lambda(df,dg) + fE(g) - gE(f)\ , \quad \text{for all} \quad f,g\in C^\infty(M,\mathbb{R}). \tag{2.2}$$

The Jacobi bracket $\{\ ,\ \}$ is skew-symmetric, satisfies the Jacobi identity and, in addition, we have

$$\text{support}\{f,g\} \subseteq (\text{support } f)\cap(\text{support } g)\ , \quad \text{for all} \quad f,g\in C^\infty(M,\mathbb{R}).$$

Thus, the space $C^\infty(M,\mathbb{R})$ endowed with the Jacobi bracket is *a local Lie algebra* in the sense of Kirillov (see [25]). Conversely, a structure of local Lie algebra on $C^\infty(M,\mathbb{R})$ defines a Jacobi structure on M (see [21, 25]). If the vector field E identically vanishes then $\{\ ,\ \}$ is a derivation in each argument and, therefore, $\{\ ,\ \}$ defines a *Poisson bracket* on M. In this case, (2.1) reduces to $[\Lambda,\Lambda]=0$ and (M,Λ) is a *Poisson manifold.* Jacobi and Poisson manifolds were introduced by Lichnerowicz ([38, 39]).

Remark 2.1 Let (Λ, E) be a Jacobi structure on a manifold M and consider on the product manifold $M \times \mathbb{R}$ the 2-vector $\tilde{\Lambda}$ given by

$$\tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E),$$

where t is the usual coordinate on $\mathbb{R}$. Then, $\tilde{\Lambda}$ defines a Poisson structure on $M \times \mathbb{R}$ (see [39]). The manifold $M \times \mathbb{R}$ endowed with the structure $\tilde{\Lambda}$ is called the *poissonization of the Jacobi manifold* (M, Λ, E).

2.2 Examples of Jacobi manifolds

In this section, we will present some examples of Jacobi manifolds.

1. *Symplectic manifolds.*- A *symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a closed non-degenerate 2-form on M. We define a Poisson 2-vector Λ on M by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)), \tag{2.3}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X\Omega$ (see [38]).

Using the classical theorem of Darboux, around every point of M there exist canonical coordinates $(q^1, \dots, q^m, p_1, \dots, p_m)$ on M such that

$$\Omega = \sum_i dq^i \wedge dp_i, \qquad \Lambda = \sum_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

2. *Lie-Poisson structures.*- Let $(\mathfrak{g}, [\ ,\ ])$ be a real Lie algebra of dimension n with Lie bracket $[\ ,\ ]$ and denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Given two functions $f, g \in C^\infty(\mathfrak{g}^*, \mathbb{R})$, we define $\{f, g\}$ as follows. For a point $x \in \mathfrak{g}^*$, we linearize f and g, namely, we take the tangent maps $df(x)$ and $dg(x)$ at x and identify them with two elements $\hat{f}, \hat{g} \in \mathfrak{g}$. Thus, $[\hat{f}, \hat{g}] \in \mathfrak{g}$, and we define

$$\{f, g\}(x) = [\hat{f}, \hat{g}].$$

$\{\ ,\ \}$ is the so-called *Lie-Poisson bracket* on $\mathfrak{g}^*$ (see [52, 54]).

If $\bar{\Lambda}$ is the corresponding Poisson 2-vector on $\mathfrak{g}^*$ and (x_i) are global coordinates for $\mathfrak{g}^*$ obtained from a basis, we have

$$\bar{\Lambda} = \frac{1}{2} \sum_{i,j,k} c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{2.4}$$

c_{ij}^k being the structure constants of $\mathfrak{g}$.

From (2.4), it follows that

$$\mathcal{L}_A \bar{\Lambda} = -\bar{\Lambda}, \tag{2.5}$$

where A is the radial vector field on $\mathfrak{g}^*$. Note that the expression of A with respect to the coordinates (x_i) is

$$A = \sum_i x_i \frac{\partial}{\partial x_i}. \tag{2.6}$$

3. *Contact manifolds.*- Let M be a $(2m+1)$-dimensional manifold and η a 1-form on M. We say that η is a contact 1-form if $\eta \wedge (d\eta)^m \neq 0$ at every point. In such a case (M, η) is termed a *contact manifold* (see, for example, [4, 36, 39]). If (M, η) is a contact manifold, we define a 2-vector Λ and a vector E on M as follows

$$\Lambda(\alpha, \beta) = d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)), \quad E = \flat^{-1}(\eta) \tag{2.7}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X d\eta + \eta(X)\eta$. Then (M, Λ, E) is a Jacobi manifold. The vector field E is just the *Reeb vector field* of M and it is characterized by the relations

$$i_E \eta = 1, \quad i_E d\eta = 0. \tag{2.8}$$

Using the generalized Darboux theorem, we deduce that around every point of M there exist canonical coordinates $(t, q^1, \ldots, q^m, p_1, \ldots, p_m)$ such that (see [36, 39])

$$\eta = dt - \sum_i p_i dq^i, \qquad \Lambda = \sum_i \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\right) \wedge \frac{\partial}{\partial p_i}, \qquad E = \frac{\partial}{\partial t}. \tag{2.9}$$

Remark 2.2 The poissonization of a contact structure is a symplectic structure (see [39]).

4. *Locally conformal symplectic manifolds.*- An *almost symplectic manifold* is a pair (M, Ω), where M is an even dimensional manifold and Ω is a non-degenerate 2-form on M. An almost symplectic manifold is said to be *locally conformal symplectic (l.c.s.)* if for each point $x \in M$ there is an open neighborhood U such that $d(e^{-f}\Omega) = 0$, for some function $f : U \longrightarrow \mathbb{R}$ (see, for example, [21, 50]). So, $(U, = e^{-f}\Omega)$ is a symplectic manifold. If $U = M$ then M is said to be a *globally conformal symplectic (g.c.s.)* manifold. An almost symplectic manifold (M, Ω) is l.(g.)c.s. if and only if there exists a closed (exact) 1-form ω such that

$$d\Omega = \omega \wedge \Omega. \tag{2.10}$$

The 1-form ω is called the *Lee 1-form* of M. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just the symplectic manifolds.

In a similar way that for contact manifolds, we define a 2-vector Λ and a vector field E on M which are given by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)), \qquad E = \flat^{-1}(\omega)\ , \tag{2.11}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. Then (M, Λ, E) is a Jacobi manifold (see [21]). Note that

$$\mathcal{L}_E\Omega = 0. \tag{2.12}$$

Using the classical theorem of Darboux, around every point of M there exist canonical coordinates $(q^1, \ldots, q^m, p_1, \ldots, p_m)$ and a local differentiable function f such that

$$\begin{aligned} \Omega &= e^f \sum_i dq^i \wedge dp_i, & \omega &= df = \sum_i (\frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i), \\ \Lambda &= e^{-f} \sum_i (\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}), & E &= e^{-f} \sum_i (\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}). \end{aligned}$$

5. *Unit sphere of a real Lie algebra*.- Let $\mathfrak{g}$ be a real Lie algebra of dimension n with Lie bracket $[\ ,\ ]$ and let $\bar{\Lambda}$ be the Poisson 2-vector on the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$.

Suppose that $<\ ,\ >$ is a scalar product on $\mathfrak{g}$ and that g is the corresponding Riemannian metric on $\mathfrak{g}$.

Denote by $\flat_{<\,,\,>} : \mathfrak{g} \to \mathfrak{g}^*$ the linear isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by

$$\flat_{<\,,\,>}(\xi)(\eta) =< \xi, \eta >, \qquad \text{for all } \xi, \eta \in \mathfrak{g}. \tag{2.13}$$

Using this isomorphism and the Lie-Poisson structure $\bar{\Lambda}$, we can define a Poisson structure on $\mathfrak{g}$ which we also denote by $\bar{\Lambda}$.

Now, we consider the 2-vector Λ' and the vector field E' on $\mathfrak{g}$ given by

$$\Lambda' = \bar{\Lambda} - A \wedge i_\alpha\bar{\Lambda}, \qquad E' = i_\alpha\bar{\Lambda}, \tag{2.14}$$

where A is the radial vector field on $\mathfrak{g}$ and α is the 1-form defined by $\alpha(X) = g(X, A)$, for $X \in \mathfrak{X}(\mathfrak{g})$. From (2.6), we obtain that

$$\alpha = \frac{1}{2} d(\|\ ,\ \|^2), \quad \mathcal{L}_A\alpha = 2\alpha, \tag{2.15}$$

$\|\ ,\ \|^2 : \mathfrak{g} \longrightarrow \mathbb{R}$ being the real function on $\mathfrak{g}$ given by

$$\|\ ,\ \|^2(\xi) =< \xi, \xi >,$$

for all $\xi \in \mathfrak{g}$. Using (2.5) and (2.15), we deduce that

$$[A, E'] = E', \quad \mathcal{L}_{E'}\bar{\Lambda} = \frac{1}{2}[[\bar{\Lambda}, \|\ ,\ \|^2], \bar{\Lambda}] = 0. \tag{2.16}$$

Thus, the pair (Λ', E') induces a Jacobi structure on $\mathfrak{g}$. Moreover, if $S^{n-1}(\mathfrak{g})$ is the unit sphere in $\mathfrak{g}$, it follows that the restrictions Λ and E to $S^{n-1}(\mathfrak{g})$ of Λ' and E', respectively, are tangent to $S^{n-1}(\mathfrak{g})$. Therefore, the pair (Λ, E) defines a Jacobi structure on $S^{n-1}(\mathfrak{g})$ (for more details, see [40]).

If for every $\xi \in \mathfrak{g}$, we consider the function $<\xi, \ >: S^{n-1}(\mathfrak{g}) \longrightarrow \mathbb{R}$ given by

$$<\xi, \ > (\eta) =<\xi, \eta>, \tag{2.17}$$

then, from (2.4), (2.14) and (2.17), we have that

$$\{<\xi, \ >, <\eta, \ >\} =< [\xi,\eta], \ > \tag{2.18}$$

for $\xi, \eta \in \mathfrak{g}$, where $\{ \ , \ \}$ is the Jacobi bracket on $S^{n-1}(\mathfrak{g})$.

Note that if $\xi \in S^{n-1}(\mathfrak{g})$ it follows that $(d < \xi, \ >)(\xi) = 0$ and consequently

$$E(\xi) = X_{<\xi, >}(\xi).$$

This implies that the characteristic foliation of $S^{n-1}(\mathfrak{g})$ is generated by the set of hamiltonian vector fields $\{X_{<\xi, >}/\xi \in \mathfrak{g}\}$.

On the other hand, if (x^i) are global coordinates for $\mathfrak{g}$ obtained from an orthonormal basis $\{\xi_i\}_{i=1,\dots,n}$ of $\mathfrak{g}$ then

$$\Lambda' = \sum_{i,j,k,h,r} (\frac{1}{2}c^r_{hj}x^r - c^k_{ij}x^k x^i x^h)\frac{\partial}{\partial x^h} \wedge \frac{\partial}{\partial x^j}, \qquad E' = \sum_{i,j,k} c^k_{ij}x^k x^i \frac{\partial}{\partial x^j},$$

c^k_{ij} being the structure constants for $\mathfrak{g}$ with respect to the basis $\{\xi_i\}_{i=1,\dots,n}$.

Remark 2.3 Using the results of [40], we obtain that the poissonization of the Jacobi manifold $(S^{n-1}(\mathfrak{g}), \Lambda, E)$ is isomorphic to the Poisson manifold $(\mathfrak{g} - \{0\}, \bar{\Lambda}_{|\mathfrak{g}-\{0\}})$. In fact, an isomorphism between these Poisson manifolds is defined by

$$F : \mathfrak{g} - \{0\} \to S^{n-1}(\mathfrak{g}) \times \mathbb{R}, \qquad \xi \mapsto F(\xi) = (\frac{\xi}{\|\xi\|}, \ln\|\xi\|). \tag{2.19}$$

2.3 The characteristic foliation of a Jacobi manifold

Let (M, Λ, E) be a Jacobi manifold. Define a homomorphism of $C^\infty(M, \mathbb{R})$-modules $\#_\Lambda : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ by

$$(\#_\Lambda(\alpha))(\beta) = \Lambda(\alpha, \beta), \tag{2.20}$$

for $\alpha, \beta \in \Omega^1(M)$. This homomorphism can be extended to a homomorphism, which we also denote by $\#_\Lambda$, from the space $\Omega^k(M)$ onto the space $\mathcal{V}^k(M)$ by putting:

$$\#_\Lambda(f) = f, \quad \#_\Lambda(\alpha)(\alpha_1, \dots, \alpha_k) = (-1)^k \alpha(\#_\Lambda(\alpha_1), \dots, \#_\Lambda(\alpha_k)), \tag{2.21}$$

for $f \in C^\infty(M, \mathbb{R})$, $\alpha \in \Omega^k(M)$ and $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$.

Remark 2.4 *i)* If M is a contact manifold with Reeb vector field E, then $\#_\Lambda(\alpha) = -\flat^{-1}(\alpha) + \alpha(E)E$, for all $\alpha \in \Omega^1(M)$.

ii) If M is a l.c.s. manifold then $\#_\Lambda(\alpha) = -\flat^{-1}(\alpha)$, for all $\alpha \in \Omega^1(M)$.

If f is a C^∞ real-valued function on a Jacobi manifold M, the vector field X_f defined by

$$X_f = \#_\Lambda(df) + fE \tag{2.22}$$

is called the *hamiltonian vector field* associated with f. It should be noticed that the hamiltonian vector field associated with the constant function 1 is just E. A direct computation proves that (see [39, 43])

$$[X_f, X_g] = X_{\{f,g\}}, \tag{2.23}$$

which shows that the mapping

$$C^\infty(M, \mathbb{R}) \longrightarrow \mathfrak{X}(M), \qquad f \mapsto X_f$$

is a Lie algebra homomorphism.

Now, for every $x \in M$, we consider the subspace $\mathcal{F}_x$ of T_xM generated by all the hamiltonian vector fields evaluated at the point x. In other words, $\mathcal{F}_x = (\#_\Lambda)_x(T_x^*M) + \langle E_x \rangle$. Since $\mathcal{F}$ is involutive, one easily follows that $\mathcal{F}$ defines a generalized foliation in the sense of Sussmann [48], which is called the *characteristic foliation* (see [12, 21]). Moreover, the Jacobi structure of M induces a Jacobi structure on each leaf. In fact, if L is the leaf over a point x of M and $E_x \notin Im(\#_\Lambda)_x$ (or equivalently, the dimension of L is odd) then L is a contact manifold with the induced Jacobi structure. If $E_x \in Im(\#_\Lambda)_x$ (or equivalently, the dimension of L is even), L is a l.c.s. manifold (for a detailed study of the characteristic foliation, we refer to [12, 21]). If M is a Poisson manifold then, from (2.20) and (2.22), we deduce that the characteristic foliation of M is just the *canonical symplectic foliation* of M (see [52, 54]).

For a contact (respectively, l.c.s.) manifold M there exists a unique leaf of its characteristic foliation: the manifold M.

On the other hand, if $\mathfrak{g}$ is a real Lie algebra of dimension n and G is a connected Lie group with Lie algebra $\mathfrak{g}$, then the leaves of the symplectic foliation associated to the Lie-Poisson structure on $\mathfrak{g}^*$ are just the orbits of the coadjoint action $Ad^* : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ (see [52, 54]).

Moreover, if $< \, , \, >$ is a scalar product on $\mathfrak{g}$ then, under the canonical identification between $\mathfrak{g}$ and $\mathfrak{g}^*$, the coadjoint action induces an action of the Lie group G on $\mathfrak{g}$, which we will denote by $\widetilde{Ad^*}$. Thus, we can define an action of G on the unit sphere $S^{n-1}(\mathfrak{g})$ as follows:

$$\overline{Ad}^* : G \times S^{n-1}(\mathfrak{g}) \longrightarrow S^{n-1}(\mathfrak{g}), \qquad (g, \xi) \mapsto \overline{Ad}^*(g, \xi) = \frac{\widetilde{Ad^*}(g, \xi)}{\|\widetilde{Ad^*}(g, \xi)\|}. \tag{2.24}$$

Now, denote by (Λ, E) the Jacobi structure on $S^{n-1}(\mathfrak{g})$ defined in Section 2.2 and by $\xi_{S^{n-1}(\mathfrak{g})}$ the infinitesimal generator, with respect to the action $\overline{Ad}^*$, associated to $\xi \in \mathfrak{g}$. Then, using the results of [40], we deduce that $\xi_{S^{n-1}(\mathfrak{g})}$ is the hamiltonian vector field on $(S^{n-1}(\mathfrak{g}), \Lambda, E)$ associated to the function $< \xi, \, >: S^{n-1}(\mathfrak{g}) \longrightarrow \mathbb{R}$ given by (2.17), that is,

$$\xi_{S^{n-1}(\mathfrak{g})} = X_{<\xi, \, >}. \tag{2.25}$$

This fact implies that the leaves of the characteristic foliation of $S^{n-1}(\mathfrak{g})$ are just the orbits of the action $\overline{Ad}^*$ (for more details, see [40]).

2.4 Lie algebroid of a Jacobi manifold

A *Lie algebroid structure* on a differentiable vector bundle $\pi : K \longrightarrow M$ is a pair that consists of a Lie algebra structure $[\![\ ,\ ]\!]$ on the space $\Gamma(K)$ of the global cross sections of $\pi : K \longrightarrow M$ and a homomorphism of vector bundles $\varrho : K \longrightarrow TM$, the *anchor map*, such that if we also denote by $\varrho : \Gamma(K) \longrightarrow \mathfrak{X}(M)$ the homomorphism of $C^\infty(M,\mathbb{R})$-modules induced by the anchor map then:

(i) $\varrho : (\Gamma(K), [\![\ ,\ ]\!]) \longrightarrow (\mathfrak{X}(M), [\ ,\ ])$ is a Lie algebra homomorphism.

(ii) For all $f \in C^\infty(M,\mathbb{R})$ and for all $s_1, s_2 \in \Gamma(K)$ one has

$$[\![s_1, fs_2]\!] = f[\![s_1, s_2]\!] + (\varrho(s_1)(f))s_2.$$

A triple $(K, [\![\ ,\ ]\!], \varrho)$ is called *a Lie algebroid over* M (see [47, 52]).

Let (M, Λ, E) be a Jacobi manifold. In [24], the authors obtain a Lie algebroid structure on the vector bundle $J^1(M,\mathbb{R}) \cong T^*M \times \mathbb{R} \longrightarrow M$ as follows.

Consider the homomorphism of $C^\infty(M,\mathbb{R})$-modules

$$(\#_\Lambda, E) : \Gamma(J^1(M,\mathbb{R})) \cong \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M)$$

defined by

$$(\#_\Lambda, E)(\alpha, f) = \#_\Lambda(\alpha) + fE. \tag{2.26}$$

It is clear that the vector field $(\#_\Lambda, E)(\alpha, f)$ is tangent to the characteristic foliation (note that $(\#_\Lambda, E)(df, f) = X_f$). Moreover, if $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ then (see [24])

$$[\#_\Lambda(\alpha) + fE, \#_\Lambda(\beta) + gE] = \#_\Lambda(\gamma) + hE, \tag{2.27}$$

with $(\gamma, h) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ given by

$$\begin{array}{rcl} \gamma & = & \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - d(\Lambda(\alpha,\beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha\wedge\beta), \\ h & = & \alpha(\#_\Lambda(\beta)) + \#_\Lambda(\alpha)(g) - \#_\Lambda(\beta)(f) + fE(g) - gE(f). \end{array} \tag{2.28}$$

This result suggests to introduce the mapping $[\![\ ,\ ]\!]_{(\Lambda,E)} : (\Omega^1(M)\times C^\infty(M,\mathbb{R}))^2 \longrightarrow \Omega^1(M)\times C^\infty(M,\mathbb{R})$ defined by

$$[\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} = (\gamma, h). \tag{2.29}$$

This mapping gives a Lie algebra structure on $\Omega^1(M) \times C^\infty(M,\mathbb{R})$ in such a way that the triple $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ is a Lie algebroid over M (see [24]).

Remark 2.5 *i*) If $\{\ ,\ \}$ is the Jacobi bracket then the prolongation mapping

$$j^1 : (C^\infty(M,\mathbb{R}), \{\ ,\ \}) \longrightarrow (\Omega^1(M) \times C^\infty(M,\mathbb{R}), [\![\ ,\ ]\!]_{(\Lambda,E)}) \qquad f \mapsto j^1 f = (df, f) \tag{2.30}$$

is a Lie algebra homomorphism (see [24]).

ii) In the particular case when M is a Poisson manifold we recover, by projection, the usual Lie algebroid structure on the vector bundle $\pi : T^*M \longrightarrow M$ (see [2, 3, 9]).

3 Lichnerowicz-Jacobi cohomology of a Jacobi manifold

3.1 H-Chevalley-Eilenberg cohomology and Lichnerowicz-Jacobi cohomology of a Jacobi manifold

First of all, we recall the definition of the cohomology of a Lie algebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module (we will follow [52]).

Let $(\mathcal{A}, [\ ,\ ])$ be a real Lie algebra (not necessarily finite dimensional) and $\mathcal{M}$ a real vector space endowed with a $\mathbb{R}$-bilinear multiplication

$$\mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M}, \quad (a, m) \longrightarrow a.m$$

such that

$$[a_1, a_2].m = a_1.(a_2.m) - a_2.(a_1.m), \tag{3.1}$$

for $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$. In other words, $\mathcal{A}$ acts on $\mathcal{M}$ on the left. In such a case, a k-linear skew-symmetric mapping $c^k : \mathcal{A}^k \longrightarrow \mathcal{M}$ is called an *$\mathcal{M}$-valued k-cochain* . These cochains form a real vector space $C^k(\mathcal{A}; \mathcal{M})$ and the linear operator $\partial^k : C^k(\mathcal{A}; \mathcal{M}) \longrightarrow C^{k+1}(\mathcal{A}; \mathcal{M})$ given by

$$\begin{aligned} (\partial^k c^k)(a_0, \cdots, a_k) &= \sum_{i=0}^{k} (-1)^i a_i.c^k(a_0, \cdots, \widehat{a_i}, \cdots, a_k) + \\ &\quad + \sum_{i<j} (-1)^{i+j} c^k([a_i, a_j], a_0, \cdots, \widehat{a_i}, \cdots, \widehat{a_j}, \cdots, a_k) \end{aligned} \tag{3.2}$$

defines a coboundary since $\partial^{k+1} \circ \partial^k = 0$. Hence we have the corresponding cohomology spaces

$$H^k(\mathcal{A}; \mathcal{M}) = \frac{\ker\{\partial^k : C^k(\mathcal{A}; \mathcal{M}) \to C^{k+1}(\mathcal{A}; \mathcal{M})\}}{\mathrm{Im}\{\partial^{k-1} : C^{k-1}(\mathcal{A}; \mathcal{M}) \to C^k(\mathcal{A}; \mathcal{M})\}}.$$

This cohomology is called *the cohomology of the Lie algebra $\mathcal{A}$ with coefficients in $\mathcal{M}$, or relative to the given representation of $\mathcal{A}$ on $\mathcal{M}$.*

The *Chevalley-Eilenberg cohomology* of a Lie algebra $(\mathcal{A}, [\ ,\ ])$ is just the cohomology of $\mathcal{A}$ relative to the representation of $\mathcal{A}$ on itself given by

$$a.m = [a, m].$$

Now, let (M, Λ, E) be a Jacobi manifold with Jacobi bracket $\{\ ,\ \}$. We consider the cohomology of the Lie algebra $(C^\infty(M, \mathbb{R}), \{\ ,\ \})$ relative to the representation defined by the hamiltonian vector fields, that is,

$$C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}), \qquad (f, g) \longrightarrow X_f(g). \tag{3.3}$$

This cohomology is denoted by $H^*_{HCE}(M)$ and it was called the *H-Chevalley-Eilenberg cohomology* associated to M (see [30, 32, 33, 34]). Explicitly, if $C^k_{HCE}(M)$ is the real vector space of the k-linear skew-symmetric mappings $c^k : C^\infty(M,\mathbb{R}) \times \ldots^{(k} \ldots \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ then

$$H^k_{HCE}(M) = \frac{\ker\{\partial_H : C^k_{HCE}(M) \to C^{k+1}_{HCE}(M)\}}{\mathrm{Im}\{\partial_H : C^{k-1}_{HCE}(M) \to C^k_{HCE}(M)\}},$$

where $\partial_H : C^r_{HCE}(M) \longrightarrow C^{r+1}_{HCE}(M)$ is the linear differential operator defined by

$$\begin{aligned}(\partial_H c^r)(f_0, \cdots, f_r) &= \sum_{i=0}^{r}(-1)^i X_{f_i}(c^r(f_0, \cdots, \widehat{f}_i, \cdots, f_r))+ \\ &+\sum_{i<j}(-1)^{i+j} c^r(\{f_i, f_j\}, f_0, \cdots, \widehat{f}_i, \cdots, \widehat{f_j}, \cdots, f_r)\end{aligned} \tag{3.4}$$

for $c^r \in C^r_{HCE}(M)$ and $f_0, \ldots, f_r \in C^\infty(M,\mathbb{R})$.

Note that for a Poisson manifold, $H^*_{HCE}(M)$ is the *Chevalley-Eilenberg cohomology* of the Lie algebra $(C^\infty(M,\mathbb{R}), \{\ ,\ \})$ (see [38]). However, for arbitrary Jacobi manifolds the Chevalley-Eilenberg cohomology (which is defined with respect to the representation given by the Jacobi bracket [39]) does not coincide in general with the H-Chevalley-Eilenberg cohomology defined above.

An interesting subcomplex of the H-Chevalley-Eilenberg complex is the complex of the 1-differentiable cochains.

A k-cochain $c^k \in C^k_{HCE}(M)$ is said to be 1-*differentiable* if it is defined by a k-linear skew-symmetric differential operator of order 1. We can identify the space $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ with the space of all 1-differentiable k-cochains $C^k_{HCE-1diff}(M)$ as follows (see, for instance, [38]): define $j^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow C^k_{HCE}(M)$ the monomorphism given by

$$j^k(P,Q)(f_1, \cdots, f_k) = P(df_1, \ldots, df_k) + \sum_{q=1}^{k}(-1)^{q+1} f_q Q(df_1, \cdots, \widehat{df_q}, \cdots, df_k). \tag{3.5}$$

Then, $j^k(\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)) = C^k_{HCE1-diff}(M)$ which implies that the spaces $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and $C^k_{HCE1-diff}(M)$ are isomorphic.

On the other hand, if $\tilde{P} \in C^k_{HCE1-diff}(M)$ then $\partial_H \tilde{P} \in C^{k+1}_{HCE1-diff}(M)$. Thus, we have the corresponding subcomplex $(C^*_{HCE1-diff}(M), \partial_{H|C^*_{HCE1-diff}(M)})$ of the H-Chevalley-Eilenberg complex whose cohomology $H^*_{HCE1-diff}(M)$ will be called the 1-*differentiable H-Chevalley-Eilenberg cohomology of* M (see [33, 34]). Moreover, using (3.4), (3.5) and the properties of the Schouten-Nijenhuis bracket, we can prove that

$$\partial_H(j^k(P,Q)) = j^{k+1}(\sigma(P,Q)), \tag{3.6}$$

where

$$\sigma(P,Q) = (-[\Lambda, P] + kE \wedge P + \Lambda \wedge Q, [\Lambda, Q] - (k-1)E \wedge Q + [E,P]). \tag{3.7}$$

The last equation defines a mapping $\sigma : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow \mathcal{V}^{k+1}(M) \oplus \mathcal{V}^k(M)$ which is in fact a differential operator that verifies $\sigma^2 = 0$. Thus, we have a complex $(\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma)$ whose cohomology will be called the *Lichnerowicz-Jacobi cohomology (LJ-cohomology) of* M and denoted by $H^*_{LJ}(M, \Lambda, E)$ or simply by $H^*_{LJ}(M)$ if there is not danger of confusion (see [33, 34]). This cohomology is a generalization of the Lichnerowicz-Jacobi cohomology introduced in [30, 31, 32]. In fact, the former one is the cohomology of the subcomplex of the pairs $(P, 0)$, where P is invariant by E. For this reason, we retain the name.

Notice that the mappings $j^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow C^k_{HCE}(M)$ given by (3.5) induce an isomorphism between the complexes

$$(\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma) \quad \text{and} \quad (C^*_{HCE1-diff}(M), (\partial_H)_{|C^*_{HCE1-diff}(M)})$$

and therefore the corresponding cohomologies are isomorphic.

Remark 3.1 If $\tilde{\sigma}$ denotes the cohomology operator in the 1-differentiable Chevalley-Eilenberg subcomplex then (see [39])

$$\tilde{\sigma}(P, Q) = (-[\Lambda, P] + (k-1)E \wedge P + \Lambda \wedge Q, [\Lambda, Q] - (k-2)E \wedge Q + [E, P]),$$

for $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. Thus, from (3.7), we deduce that the 1-differentiable H-Chevalley-Eilenberg cohomology (that is, the LJ-cohomology) does not coincide in general with the 1-differentiable Chevalley-Eilenberg cohomology.

To end this subsection, we will present another description of the LJ-cohomology in terms of the Lie algebroid associated with the Jacobi manifold (see [33, 34, 53]).

Let $(K, [\![\ ,\ ]\!], \varrho)$ be a Lie algebroid over M. For $r \geq 0$, let $\Gamma(\Lambda^r K^*)$ be the space of the smooth sections of $\Lambda^r K^*$, that is, $\Gamma(\Lambda^r K^*)$ is the space of $C^\infty(M, \mathbb{R})$-linear skew-symmetric mappings $\xi^r : \Gamma(K) \times \dots^{(r} \dots \times \Gamma(K) \longrightarrow C^\infty(M, \mathbb{R})$. Define

$$\tilde{\partial}^r : \Gamma(\Lambda^r K^*) \longrightarrow \Gamma(\Lambda^{r+1} K^*)$$

by

$$\begin{aligned}(\tilde{\partial}^r \xi^r)(s_0, \cdots, s_r) \quad &= \quad \sum_{i=0}^{r} (-1)^i \varrho(s_i)(\xi^r(s_0, \cdots, \widehat{s_i}, \cdots, s_r)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \xi^r([\![s_i, s_j]\!], s_0, \cdots, \widehat{s_i}, \cdots, \widehat{s_j}, \cdots, s_r).\end{aligned} \tag{3.8}$$

The operator $\tilde{\partial}^r$ satisfies $\tilde{\partial}^{r+1} \circ \tilde{\partial}^r = 0$. The corresponding cohomology is called the *Lie algebroid cohomology of* K *with trivial coefficients* (see [41]).

Remark 3.2 Consider the representation of the Lie algebra $(\Gamma(K), [\![\ ,\ ]\!])$ onto the space $C^\infty(M, \mathbb{R})$ given by $s.f = \varrho(s)(f)$, for all $s \in \Gamma(K)$ and $f \in C^\infty(M, \mathbb{R})$, and denote by $(C^*(\Gamma(K); C^\infty(M, \mathbb{R})), \partial)$ the corresponding differential complex. Then, the Lie algebroid cohomology of K with trivial coefficients is the one of the subcomplex of $(C^*(\Gamma(K); C^\infty(M, \mathbb{R})), \partial)$ consisting of the cochains which are $C^\infty(M, \mathbb{R})$-linear.

Now, let (M, Λ, E) be a Jacobi manifold of dimension n and $(J^1(M,\mathbb{R}), [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ the Lie algebroid over M (see Section 2.4). Then, the LJ-cohomology of M is just the Lie algebroid cohomology of $J^1(M,\mathbb{R})$ with trivial coefficients. In fact, the homomorphisms of $C^\infty(M,\mathbb{R})$-modules $\tilde{j^k} : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow \Gamma(\Lambda^k J^1(M,\mathbb{R})^*)$ defined by

$$\tilde{j^k}(P,Q)((\alpha_1, f_1), \cdots, (\alpha_k, f_k)) = P(\alpha_1, \cdots, \alpha_k) + \sum_{q=1}^{k} (-1)^{q+1} f_q Q(\alpha_1, \cdots, \widehat{\alpha_q}, \cdots, \alpha_k)$$

for $(P,Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and $(\alpha_1, f_1), \cdots, (\alpha_k, f_k) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$, induce an isomorphism between the complexes $(\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma)$ and $(\bigoplus_{k=1}^{n+1} \Gamma(\Lambda^k J^1(M,\mathbb{R})^*), \tilde{\partial}^*)$ (for more details, see [33, 34, 53]).

3.2 Lichnerowicz-Jacobi cohomology and conformal changes of Jacobi structures

In this section, we will prove that the LJ-cohomology is invariant under conformal changes.

First, we will recall some definitions and results related with the theory of Lie algebroids which will be useful in the sequel (we will follow [41]).

Suppose that the pair $([\![\ ,]\!], \varrho)$ (respectively, $([\![\ ,]\!]', \varrho')$) is a Lie algebroid structure on the vector bundle $\pi : K \longrightarrow M$ (respectively, $\pi' : K' \longrightarrow M$). An isomorphism between the Lie algebroids $(K, [\![\ ,]\!], \varrho)$ and $(K', [\![\ ,]\!]', \varrho')$ is an isomorphism of vector bundles $\phi : K \longrightarrow K'$ such that if we denote by $\phi_1 : \Gamma(K) \longrightarrow \Gamma(K')$ the isomorphism of $C^\infty(M,\mathbb{R})$-modules induced by $\phi : K \longrightarrow K'$ then:

(i) $\varrho' \circ \phi = \varrho$.

(ii) For all $s_1, s_2 \in \Gamma(K)$, $\phi_1[\![s_1, s_2]\!] = [\![\phi_1(s_1), \phi_1(s_2)]\!]'$, that is, $\phi_1 : \Gamma(K) \longrightarrow \Gamma(K')$ is a Lie algebra homomorphism.

Assume that $\phi : K \longrightarrow K'$ is an isomorphism between the Lie algebroids $(K, [\![\ ,]\!], \varrho)$ and $(K', [\![\ ,]\!]', \varrho')$. Then, we can consider the isomorphism of $C^\infty(M,\mathbb{R})$-modules $\phi^r : \Gamma(\Lambda^r(K')^*) \longrightarrow \Gamma(\Lambda^r K^*)$ given by

$$\phi^r(\xi^r)(s_1, \ldots, s_r) = \xi^r(\phi_1(s_1), \ldots, \phi_1(s_r)) \tag{3.9}$$

for $\xi^r \in \Gamma(\Lambda^r(K')^*)$ and $s_1, \ldots, s_r \in \Gamma(K)$. A direct computation, using (3.8), proves that

$$\tilde{\partial}^k \circ \phi^k = \phi^{k+1} \circ (\tilde{\partial}')^k, \tag{3.10}$$

where $\tilde{\partial}^*$ (respectively, $(\tilde{\partial}')^*$) is the cohomology operator induced by the Lie algebroid structure $([\![\ ,\ ]\!], \varrho)$ (respectively, $([\![\ ,\ ]\!]', \varrho')$). Therefore, the Lie algebroids cohomologies of K and K' with trivial coefficients are isomorphic.

Now, let (Λ, E) be a Jacobi structure on M. A *conformal change* of (Λ, E) is a new Jacobi structure (Λ_a, E_a) on M defined by

$$\Lambda_a = a\Lambda, \qquad E_a = X_a = \#_\Lambda(da) + aE, \tag{3.11}$$

a being a positive C^∞ real-valued function on M (see [12, 21]). We remark that $(\Lambda, E) = ((\Lambda_a)_{\frac{1}{a}}, (E_a)_{\frac{1}{a}})$. Moreover, we have the following

Theorem 3.3 *Let (M, Λ, E) be a Jacobi manifold and (Λ_a, E_a) a conformal change of the Jacobi structure (Λ, E). Then,*

$$H^k_{LJ}(M, \Lambda, E) \cong H^k_{LJ}(M, \Lambda_a, E_a),$$

for all k.

Proof: We define the isomorphism of vector bundles $\phi : T^*M \times \mathbb{R} \longrightarrow T^*M \times \mathbb{R}$ by

$$\phi(\alpha_x, \lambda) = (\frac{1}{a(x)}\alpha_x + \lambda d(\frac{1}{a})(x), \frac{\lambda}{a(x)}) \tag{3.12}$$

for $\alpha_x \in T^*_xM$ and $\lambda \in \mathbb{R}$. Note that the isomorphism of $C^\infty(M, \mathbb{R})$-modules $\phi_1 : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \longrightarrow \Omega^1(M) \times C^\infty(M, \mathbb{R})$ induced by ϕ is given by

$$\phi_1(\alpha, f) = (\frac{1}{a}\alpha + fd(\frac{1}{a}), \frac{f}{a}) = (\frac{1}{a}\alpha - \frac{f}{a^2}da, \frac{f}{a}) \tag{3.13}$$

for all $(\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$.

A direct computation, using (2.26), (2.28), (2.29), (3.11) and (3.13), proves that

$$(\#_{\Lambda_a}, E_a) \circ \phi = (\#_\Lambda, E), \quad \phi_1[\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} = [\![\phi_1(\alpha, f), \phi_1(\beta, g)]\!]_{(\Lambda_a,E_a)}$$

for all $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$. Thus, ϕ defines an isomorphism between the Lie algebroids $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ and $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda_a,E_a)}, (\#_{\Lambda_a}, E_a))$ associated with the Jacobi structures (Λ, E) and (Λ_a, E_a), respectively. Therefore, from the results in Section 3.1, it follows that

$$H^k_{LJ}(M, \Lambda, E) \cong H^k_{LJ}(M, \Lambda_a, E_a),$$

for all k. □

Finally, using Theorem 3.3, we deduce the result announced at the beginning of this section

Corollary 3.4 *The LJ-cohomology is invariant under conformal changes of the Jacobi structure.*

3.3 Lichnerowicz-Jacobi cohomology of a Poisson manifold

Let (M, Λ) be a Poisson manifold and σ the LJ-cohomology operator. Using (3.7), we obtain that

$$\sigma(P, Q) = (-[\Lambda, P] + \Lambda \wedge Q, [\Lambda, Q]), \tag{3.14}$$

for $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$.

Denote by $\bar{\sigma}$ the cohomology operator of the subcomplex of the pairs $(P, 0)$. Under the canonical identification $\mathcal{V}^k(M) \oplus \{0\} \cong \mathcal{V}^k(M)$, we have that

$$\bar{\sigma}(P) = -[\Lambda, P]. \tag{3.15}$$

The cohomology of the complex $(\mathcal{V}^*(M), \bar{\sigma})$ is called the *Lichnerowicz-Poisson cohomology* (LP-cohomology) of M and denoted by $H^*_{LP}(M, \Lambda)$ or simply by $H^*_{LP}(M)$ if there is not danger of confusion (see [38, 52]). Note that $\bar{\sigma}(\Lambda) = 0$ and thus Λ is a 2-cocycle in the LP-complex of M. Therefore, we can define the homomorphism $L^k : H^k_{LP}(M) \longrightarrow H^{k+2}_{LP}(M)$ by

$$L^k[P] = [P \wedge \Lambda]$$

for all $[P] \in H^k_{LP}(M)$.

In [38] (see also [37]), Lichnerowicz has exhibited the relation between the LJ-cohomology (the 1-differentiable Chevalley-Eilenberg cohomology in his terminology) and the LP-cohomology of a Poisson manifold. In fact, if $\dim H^k_{LP}(M) < \infty$, for all k, we have that

$$H^k_{LJ}(M) \cong \frac{H^k_{LP}(M)}{Im\, L^{k-2}} \oplus \ker\, L^{k-1}\ . \tag{3.16}$$

Next, we will obtain some consequences of the results of Lichnerowicz and of another authors about the LJ-cohomology of symplectic and Lie-Poisson structures.

3.3.1 Symplectic structures

Let (M, Ω) be a symplectic manifold of dimension $2m$. Denote by Λ the Poisson 2-vector and by $\#_\Lambda : \Omega^k(M) \longrightarrow \mathcal{V}^k(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules given by (2.20) and (2.21). Since, in this case, $\#_\Lambda$ is an isomorphism of $C^\infty(M, \mathbb{R})$-modules and

$$\#_\Lambda(d\alpha) = -\bar{\sigma}(\#_\Lambda(\alpha)) \tag{3.17}$$

for all $\alpha \in \Omega^k(M)$ (see [38, 52]), it follows that $\#_\Lambda$ induces an isomorphism between the de Rham cohomology of M, $H^*_{dR}(M)$, and the LP-cohomology. Thus, $H^k_{dR}(M) \cong H^k_{LP}(M)$.

Under this identification and since $\#_\Lambda(\Omega) = \Lambda$, the homomorphism $L^k : H^k_{LP}(M) \cong H^k_{dR}(M) \longrightarrow H^{k+2}_{LP}(M) \cong H^{k+2}_{dR}(M)$ is given by

$$L^k([\alpha]) = [\alpha \wedge \Omega], \tag{3.18}$$

for all $[\alpha] \in H^k_{dR}(M)$ and $0 \le k \le 2m$.

From (3.16), we have that

$$H^k_{LJ}(M) \cong \frac{H^k_{dR}(M)}{\mathrm{Im} L^{k-2}} \oplus \ker L^{k-1}. \tag{3.19}$$

Therefore, if $b_r(M)$ is the r-th Betti number of M, we obtain

$$\begin{array}{rcl} \dim H^k_{LJ}(M) & \le & b_k(M) + b_{k-1}(M), \\ \dim H^k_{LJ}(M) & \ge & \max\{b_k(M) - b_{k-2}(M),\ b_{k-1}(M) - b_{k+1}(M)\}. \end{array} \tag{3.20}$$

Next, we will discuss the behaviour of some examples of symplectic manifolds with respect to the inequalities (3.20).

Exact symplectic manifolds.- If Ω is an exact 2-form then the homomorphisms L^k are null and, from (3.19), it follows that $H^k_{LJ}(M) \cong H^k_{dR}(M) \oplus H^{k-1}_{dR}(M)$ (see [37]). Consequently,

$$\dim H^k_{LJ}(M) = b_k(M) + b_{k-1}(M).$$

In particular, the dimension of $H^k_{LJ}(M)$ is a topological invariant of M, for all k.

Lefschetz symplectic manifolds.- A symplectic manifold (M, Ω) of dimension $2m$ is said to be a *Lefschetz symplectic manifold* if it satisfies the strong Lefschetz theorem, that is, if for every k, $0 \le k \le m$, the homomorphism

$$\Delta^k = L^{k+2(m-k-1)} \circ \ldots \circ L^{k+2} \circ L^k : H^k_{dR}(M) \longrightarrow H^{2m-k}_{dR}(M), \quad [\alpha] \mapsto \Delta^k([\alpha]) = [\alpha \wedge \Omega^{m-k}]$$

is an isomorphism.

If (M, Ω) is a Lefschetz symplectic manifold then it is easy to prove that:

(i) L^k is a monomorphism, for $k \le m-1$.

(ii) L^k is an epimorphism, for $k \ge m-1$.

Thus, we deduce that

$$\begin{array}{ll} b_k(M) - b_{k-2}(M) \le 0, & \text{for} \quad k \ge m+1 \\ b_{k-1}(M) - b_{k+1}(M) \le 0, & \text{for} \quad k \le m. \end{array}$$

Moreover, using (3.19), we have that

$$\begin{array}{ll} H^k_{LJ}(M) \cong \dfrac{H^k_{dR}(M)}{\mathrm{Im} L^{k-2}}, & \text{for } k \le m, \\ H^k_{LJ}(M) \cong \ker L^{k-1}, & \text{for } k \ge m+1, \end{array}$$

which implies that

$$\begin{array}{ll} \dim H^k_{LJ}(M) = b_k(M) - b_{k-2}(M), & \text{for } k \le m, \\ \dim H^k_{LJ}(M) = b_{k-1}(M) - b_{k+1}(M), & \text{for } k \ge m+1. \end{array}$$

Therefore, the dimension of $H^k_{LJ}(M)$ is a topological invariant of M, for all k.

Remark 3.5 *i*) A manifold M endowed with a complex structure J is said to be *Kähler* if it admits a Riemannian metric g compatible with J and such that the Kähler 2-form Ω given by

$$\Omega(X, Y) = g(X, JY),$$

is closed (see [26]). In such a case, Ω defines a symplectic structure on M. Furthermore, if M is compact then (M, Ω) is a Lefschetz symplectic manifold (see [57]).

ii) There exist examples of compact Lefschetz symplectic manifolds which do not admit Kähler structures (see [8, 15]).

Compact symplectic nilmanifolds.- Let G be a simply connected nilpotent Lie group of even dimension and let $\tilde{\Omega}$ be a left-invariant symplectic 2-form on G. Suppose that Γ is a discrete subgroup of G such that the space of right cosets $\Gamma\backslash G$ is a compact manifold. Then, the 2-form $\tilde{\Omega}$ induces a symplectic 2-form Ω on $\Gamma\backslash G$ and thus $\Gamma\backslash G$ is a compact symplectic nilmanifold.

Now, denote by $\mathfrak{g}$ the Lie algebra of G and by $H^*(\mathfrak{g})$ the cohomology of $\mathfrak{g}$ relative to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$:

$$\mathfrak{g} \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad (a, t) \mapsto a.t = 0.$$

We define the homomorphism $(L_{\mathfrak{g}})^k : H^k(\mathfrak{g}) \longrightarrow H^{k+2}(\mathfrak{g})$ by

$$(L_{\mathfrak{g}})^k[\alpha] = [\alpha \wedge \tilde{\Omega}_{\mathfrak{g}}] \tag{3.21}$$

for $[\alpha] \in H^k(\mathfrak{g})$, where $\tilde{\Omega}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ is the symplectic 2-form on $\mathfrak{g}$ induced by $\tilde{\Omega}$.

Using Nomizu's Theorem [46], we have that the canonical homomorphism $i^k : H^k(\mathfrak{g}) \longrightarrow H^k_{dR}(\Gamma\backslash G)$ is an isomorphism. Moreover, from (3.18) and (3.21), we deduce that

$$i^{k+2} \circ (L_{\mathfrak{g}})^k = L^k \circ i^k$$

for all k. Therefore (see (3.19))

$$H^k_{LJ}(\Gamma\backslash G) \cong \frac{H^k(\mathfrak{g})}{\mathrm{Im}(L_{\mathfrak{g}})^{k-2}} \oplus \ker(L_{\mathfrak{g}})^{k-1}. \tag{3.22}$$

Remark 3.6 *i*) A compact Lefschetz symplectic nilmanifold is necessarily a torus (see [1]).

ii) If the Lie group G is completely solvable then (3.22) also holds since, in such a case, the canonical homomorphism $i^k : H^k(\mathfrak{g}) \longrightarrow H^k_{dR}(\Gamma\backslash G)$ is also an isomorphism, for all k (see [22]).

Example 3.7 Let H be the *Heisenberg group* which consists of real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$. H is a simply connected nilpotent Lie group of dimension 3. Denote by G the nilpotent Lie group of dimension 4 defined by $G = H \times \mathbb{R}$.

If t is the usual coordinate on $\mathbb{R}$, a basis for the left-invariant 1-forms on G is given by

$$\{\tilde{\alpha} = dx, \tilde{\beta} = dy, \tilde{\eta} = dz - xdy, \tilde{\gamma} = dt\}.$$

We have that

$$d\tilde{\alpha} = d\tilde{\beta} = 0, \quad d\tilde{\eta} = -\tilde{\alpha} \wedge \tilde{\beta}, \quad d\tilde{\gamma} = 0. \tag{3.23}$$

Thus,

$$\tilde{\Omega} = \tilde{\alpha} \wedge \tilde{\eta} + \tilde{\beta} \wedge \tilde{\gamma} \tag{3.24}$$

is a left-invariant symplectic 2-form on G.

On the other hand, if $\bar{\Gamma}$ is the subgroup of H consisting of those matrices whose entries are integers then $\Gamma = \bar{\Gamma} \times \mathbb{Z}$ is a discrete subgroup of G and the space of right cosets $\Gamma \backslash G = (\bar{\Gamma} \backslash H) \times S^1$ is a compact nilmanifold. In fact, $\Gamma \backslash G$ is the *Kodaira-Thurston manifold* (see [27, 49]).

The 1-forms $\tilde{\alpha}, \tilde{\beta}, \tilde{\eta}$ and $\tilde{\gamma}$ all descend to 1-forms α, β, η and γ on $\Gamma \backslash G$. Moreover, using (3.23) and Nomizu's Theorem, it follows that

$$\begin{aligned} H^0(\mathfrak{g}) \cong H^0_{dR}(\Gamma \backslash G) &= \; < \{1\} >, \quad H^1(\mathfrak{g}) \cong H^1_{dR}(\Gamma \backslash G) = < \{[\alpha], [\beta], [\gamma]\} >, \\ H^2(\mathfrak{g}) \cong H^2_{dR}(\Gamma \backslash G) &= \; < \{[\alpha \wedge \eta], [\alpha \wedge \gamma], [\beta \wedge \eta], [\beta \wedge \gamma]\} >, \\ H^3(\mathfrak{g}) \cong H^3_{dR}(\Gamma \backslash G) &= \; < \{[\alpha \wedge \beta \wedge \eta], [\alpha \wedge \eta \wedge \gamma], [\beta \wedge \eta \wedge \gamma]\} >, \\ H^4(\mathfrak{g}) \cong H^4_{dR}(\Gamma \backslash G) &= \; < \{[\alpha \wedge \beta \wedge \eta \wedge \gamma]\} > . \end{aligned} \tag{3.25}$$

Therefore,

$$b_0(\Gamma \backslash G) = b_4(\Gamma \backslash G) = 1, \quad b_1(\Gamma \backslash G) = b_3(\Gamma \backslash G) = 3, \quad b_2(\Gamma \backslash G) = 4. \tag{3.26}$$

Now, from (3.21), (3.22) and (3.25), we deduce that

$$\begin{aligned} &\dim H^0_{LJ}(\Gamma \backslash G) = \dim H^5_{LJ}(\Gamma \backslash G) = 1, \quad \dim H^2_{LJ}(\Gamma \backslash G) = 4 \\ &\dim H^1_{LJ}(\Gamma \backslash G) = \dim H^4_{LJ}(\Gamma \backslash G) = 3, \quad \dim H^3_{LJ}(\Gamma \backslash G) = 5. \end{aligned}$$

Consequently (see (3.26)), we have

$$\max\{b_k(\Gamma \backslash G) - b_{k-2}(\Gamma \backslash G), b_{k-1}(\Gamma \backslash G) - b_{k+1}(\Gamma \backslash G)\} < \dim H^k_{LJ}(M) < b_k(\Gamma \backslash G) + b_{k-1}(\Gamma \backslash G)$$

for $k = 2, 3$.

3.3.2 Lie-Poisson structures

Let (M,Λ) be an *exact Poisson manifold*, that is, there exists a vector field X on M such that

$$\Lambda = \bar{\sigma}X = -\mathcal{L}_X\Lambda.$$

In [38], Lichnerowicz proved that, under this condition, we have

$$H^k_{LJ}(M) \cong H^k_{LP}(M) \oplus H^{k-1}_{LP}(M),$$

for all k.

Now, suppose that $\mathfrak{g}$ is a real Lie algebra of dimension n and consider the Lie-Poisson structure $\bar{\Lambda}$ on $\mathfrak{g}^*$ (see Section 2.2). Using (2.5), it follows that $(\mathfrak{g}^*,\bar{\Lambda})$ is an exact Poisson manifold. Thus,

$$H^k_{LJ}(\mathfrak{g}^*) \cong H^k_{LP}(\mathfrak{g}^*) \oplus H^{k-1}_{LP}(\mathfrak{g}^*). \tag{3.27}$$

On the other hand, if $\mathfrak{g}$ is the Lie algebra of a compact Lie group, in [19] the authors prove that

$$H^k_{LP}(\mathfrak{g}^*) \cong H^k(\mathfrak{g}) \otimes Inv, \tag{3.28}$$

where Inv is the algebra of all *Casimir functions* on $\mathfrak{g}^*$, that is,

$$Inv = \{f \in C^\infty(\mathfrak{g}^*,\mathbb{R})/X_f = 0\}.$$

Therefore, from (3.27) and (3.28), we conclude that for the Lie algebra $\mathfrak{g}$ of a compact Lie group

$$H^k_{LJ}(\mathfrak{g}^*) \cong (H^k(\mathfrak{g}) \otimes Inv) \oplus (H^{k-1}(\mathfrak{g}) \otimes Inv). \tag{3.29}$$

3.3.3 A quadratic Poisson structure

Let Λ be the quadratic Poisson structure on $\mathbb{R}^2$ defined by

$$\Lambda = xy\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\ ,$$

where (x,y) stand for the usual coordinates on $\mathbb{R}^2$.

The restriction of Λ to the open subset $\mathbb{R}^2 - (\{(x,0)/x \in \mathbb{R}\} \cup \{(0,y)/y \in \mathbb{R}\})$ is a symplectic structure and the points $(x,0),(0,y)$ with $x,y \in \mathbb{R}$, are singular points. Moreover, $(\mathbb{R}^2,\Lambda)$ is not an exact Poisson manifold and we have (see [45])

$$H^0_{LP}(\mathbb{R}^2,\Lambda) \cong \mathbb{R},\ H^1_{LP}(\mathbb{R}^2,\Lambda) \cong \mathbb{R}^2,\ H^2_{LP}(\mathbb{R}^2,\Lambda) \cong \mathbb{R}^2\ . \tag{3.30}$$

Using these facts and (3.16) we deduce

$$H^0_{LJ}(\mathbb{R}^2,\Lambda,0) \cong \mathbb{R},\ H^1_{LJ}(\mathbb{R}^2,\Lambda,0) \cong \mathbb{R}^2,\ H^2_{LJ}(\mathbb{R}^2,\Lambda,0) \cong \mathbb{R}^3,\ H^3_{LJ}(\mathbb{R}^2,\Lambda,0) \cong \mathbb{R}^2. \tag{3.31}$$

3.4 Lichnerowicz-Jacobi cohomology of a contact manifold

In this section we will study the LJ-cohomology of a contact manifold.

First, we will obtain a general result for Jacobi manifolds which relates the de Rham cohomology and the LJ-cohomology.

Let (M, Λ, E) be a Jacobi manifold. Denote by $\#_\Lambda : \Omega^k(M) \to \mathcal{V}^k(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules given by (2.20) and (2.21). Then, we have (see [31, 32]):

$$\mathcal{L}_E(\#_\Lambda(\alpha)) = \#_\Lambda(\mathcal{L}_E\alpha), \qquad -[\Lambda, \#_\Lambda(\alpha)] + kE \wedge \#_\Lambda(\alpha) = -\#_\Lambda(d\alpha) + \#_\Lambda(i_E\alpha) \wedge \Lambda, \quad (3.32)$$

for all $\alpha \in \Omega^k(M)$. Using (2.1), (3.7) and (3.32), we deduce the following

Proposition 3.8 *Let (M, Λ, E) be a Jacobi manifold and $\tilde{F}^k : \Omega^k(M) \oplus \Omega^{k-1}(M) \longrightarrow \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by*

$$\tilde{F}^k(\alpha, \beta) = (\#_\Lambda(\alpha) + E \wedge \#_\Lambda(\beta), -\#_\Lambda(i_E\alpha) + E \wedge \#_\Lambda(i_E\beta)) \quad (3.33)$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. Then, the homomorphisms $\tilde{F}^k$ induce a homomorphism of complexes

$$\tilde{F} : (\Omega^*(M), -d) \oplus (\Omega^{*-1}(M), d) \longrightarrow (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma).$$

Thus, if $H^*_{dR}(M)$ is the de Rham cohomology of M, we have the corresponding homomorphism in cohomology

$$\tilde{F} : H^*_{dR}(M) \oplus H^{*-1}_{dR}(M) \longrightarrow H^*_{LJ}(M).$$

Now, let (M, η) be a contact manifold and (Λ, E) its associated Jacobi structure. Denote by $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X(d\eta) + \eta(X)\eta$. The isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ can be extended to a mapping, which we also denote by $\flat$, from the space $\mathcal{V}^k(M)$ onto the space $\Omega^k(M)$ by putting:

$$\flat(X_1 \wedge \ldots \wedge X_k) = \flat(X_1) \wedge \ldots \wedge \flat(X_k)$$

for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$. This extension is also an isomorphism of $C^\infty(M, \mathbb{R})$-modules. In fact, it follows that

$$\#_\Lambda\alpha = (-1)^k \flat^{-1}(\alpha) + E \wedge \#_\Lambda(i_E\alpha), \quad (3.34)$$

for $\alpha \in \Omega^k(M)$ (see [32]). Moreover, we have

Theorem 3.9 *Let (M, η) be a contact manifold of dimension $2m + 1$. Then, the homomorphism*

$$\tilde{F}^k : H^k_{dR}(M) \oplus H^{k-1}_{dR}(M) \to H^k_{LJ}(M)$$

is an isomorphism for all k. Thus, $H^k_{LJ}(M) \cong H^k_{dR}(M) \oplus H^{k-1}_{dR}(M)$.

Proof: Using (3.33), (3.34) and the fact that $i_E \circ \flat = \flat \circ i_\eta$, we deduce that the homomorphism of $C^\infty(M, \mathbb{R})$-modules $\tilde{G}^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow \Omega^k(M) \oplus \Omega^{k-1}(M)$ given by

$$\tilde{G}^k(P, Q) = ((-1)^k(\flat(P) + \eta \wedge \flat(Q) - \eta \wedge \flat(i_\eta P)), (-1)^{k-1}(\flat(i_\eta P) - \eta \wedge \flat(i_\eta Q)))$$

is just the inverse homomorphism of $\tilde{F}^k : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. Consequently,

$$\tilde{F} : (\Omega^*(M), -d) \oplus (\Omega^{*-1}(M), d) \to (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma)$$

is an isomorphism of complexes. □

Remark 3.10 In [37], Lichnerowicz proved that the 1-differentiable Chevalley-Eilenberg cohomology of a contact manifold is trivial (compare this result with Theorem 3.9).

3.5 Lichnerowicz-Jacobi cohomology of a locally conformal symplectic manifold

In this section, we will study the LJ-cohomology of a l.c.s. manifold.

We will distinguish the two following cases:

i) The particular case of a g.c.s. manifold: Let (M, Ω) be a g.c.s. manifold with Lee 1-form ω and let (Λ, E) be the associated Jacobi structure. Then, there exists a C^∞ real-valued function f on M such that $\omega = df$ and the 2-form $\bar{\Omega} = e^{-f}\Omega$ is symplectic. Denote by $\bar{\Lambda}$ the Poisson 2-vector on M associated with the symplectic 2-form $\bar{\Omega}$. Using (2.3), (2.11) and Remark 2.4, we deduce that

$$\Lambda = e^{-f}\bar{\Lambda}, \quad E = \#_{\bar{\Lambda}}(d(e^{-f})).$$

Thus, the Jacobi structure (Λ, E) is a conformal change of the Poisson structure $\bar{\Lambda}$ (see (3.11)). Therefore, from (3.19) and Theorem 3.3, we obtain

Theorem 3.11 *Let (M, Ω) be a g.c.s. manifold of finite type with Lee 1-form $\omega = df$. Then,*

$$H^k_{LJ}(M) \cong \frac{H^k_{dR}(M)}{\mathrm{Im}\bar{L}^{k-2}} \oplus \ker \bar{L}^{k-1},$$

for all k, where $H^*_{dR}(M)$ is the de Rham cohomology of M and $\bar{L}^r : H^r_{dR}(M) \longrightarrow H^{r+2}_{dR}(M)$ is the homomorphism defined by

$$\bar{L}^r[\alpha] = [e^{-f}\alpha \wedge \Omega],$$

for all $[\alpha] \in H^r_{dR}(M)$.

Remark 3.12 Relation (3.19) follows directly from Theorem 3.11.

Example 3.13 Let (N, η) be a contact manifold of finite type. Consider on the product manifold $M = N \times \mathbb{R}$ the 2-form Ω given by

$$\Omega = (pr_1)^*(d\eta) - (pr_2)^*(dt) \wedge (pr_1)^*(\eta), \tag{3.35}$$

where t is the usual coordinate on $\mathbb{R}$ and pr_i $(i = 1, 2)$ are the canonical projections of M onto the first and second factor, respectively. Then, (M, Ω) is a g.c.s. manifold with Lee 1-form $\omega = (pr_2)^*(dt)$. Moreover, in this case, the symplectic 2-form $\bar{\Omega} = e^{-t}\Omega$ is exact which implies that the homomorphism $\bar{L}^r$ is null, for all r. Consequently, using Theorem 3.11, it follows that

$$H^k_{LJ}(M) \cong H^k_{dR}(M) \oplus H^{k-1}_{dR}(M) \cong H^k_{dR}(N) \oplus H^{k-1}_{dR}(N).$$

ii) The general case: Now, we will study the LJ-cohomology of an arbitrary l.c.s. manifold. First, we will obtain some results about a certain cohomology, introduced by Guedira and Lichnerowicz [21], which is associated to an arbitrary differentiable manifold endowed with a closed 1-form.

Let M be a differentiable manifoldd and ω a closed 1-form on M.

Define the differential operator d_w by (see [21])

$$d_\omega = d + e(\omega), \tag{3.36}$$

d being the exterior differential and $e(\omega)$ the operator given by

$$e(\omega)(\alpha) = \omega \wedge \alpha \tag{3.37}$$

for $\alpha \in \Omega^*(M)$.

Since ω is closed, it follows that $d^2_\omega = 0$. This result allows us to introduce the differential complex

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d_\omega} \Omega^k(M) \xrightarrow{d_\omega} \Omega^{k+1}(M) \longrightarrow \cdots$$

Denote by $H^*_\omega(M)$ the cohomology of this complex.

Proposition 3.14 *Let M be a differentiable manifold and ω a closed 1-form on M. Then:*

i) The differential complex $(\Omega^*(M), d_\omega)$ is elliptic. Thus, if M is compact the cohomology groups $H^k_\omega(M)$ have finite dimension.

ii) If ω is exact and f is a C^∞ real-valued function such that $\omega = df$ then the mapping

$$H^k_{dR}(M) \longrightarrow H^k_\omega(M), \qquad [\alpha] \mapsto [e^{-f}\alpha],$$

is an isomorphism. Therefore, $H^k_\omega(M) \cong H^k_{dR}(M)$.

Proof: i) It is easy to check that the differential operators d and d_ω have the same symbol which implies that the complex $(\Omega^*(M), d_\omega)$ is elliptic.

ii) A direct computation proves the result. $\square$

If the 1-form ω is not exact then, in general,

$$H^*_\omega(M) \not\cong H^*_{dR}(M).$$

In fact, we will show next that if M is compact and ω is non-null and parallel with respect to a Riemannian metric on M, then the cohomology $H^*_\omega(M)$ is trivial. First, we will recall some results proved by Guedira and Lichnerowicz [21] which will be useful in the sequel.

Suppose that M is a compact differentiable manifold of dimension n, that ω is a closed 1-form on M and that g is a Riemannian metric.

Consider the vector field U on M characterized by the condition

$$\omega(X) = g(X, U), \tag{3.38}$$

for all $X \in \mathfrak{X}(M)$.

Denote by δ the codifferential operator given by (see [20])

$$\delta\alpha = (-1)^{nk+n+1}(\star \circ d \circ \star)(\alpha), \tag{3.39}$$

for all $\alpha \in \Omega^k(M)$, $\star$ being the Hodge star isomorphism. Then, we define the operator $\delta_\omega : \Omega^k(M) \to \Omega^{k-1}(M)$ by (see [21])

$$\delta_\omega = \delta + i_U, \tag{3.40}$$

where i_U denotes the contraction by the vector field U, that is (see [20]),

$$i_U(\alpha) = (-1)^{nk+n}(\star \circ e(\omega) \circ \star)(\alpha), \tag{3.41}$$

for $\alpha \in \Omega^k(M)$.

Now, consider the standard scalar product $< , >$ on the space $\Omega^k(M)$:

$$< , > : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}, \qquad (\alpha, \beta) \mapsto < \alpha, \beta > = \int_M \alpha \wedge \star\beta. \tag{3.42}$$

Then, it is easy to prove that (see [21])

$$< d_\omega\alpha, \beta > = < \alpha, \delta_\omega\beta >, \tag{3.43}$$

for all $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$.

Thus, since M is compact and the complex $(\Omega^*(M), d_\omega)$ is elliptic, we obtain an orthogonal decomposition of the space $\Omega^k(M)$ as follows

$$\Omega^k(M) = \mathcal{H}^k_\omega(M) \oplus d_\omega(\Omega^{k-1}(M)) \oplus \delta_\omega(\Omega^{k+1}(M)), \tag{3.44}$$

where $\mathcal{H}^k_\omega(M) = \{\alpha \in \Omega^k(M)/d_\omega(\alpha) = 0, \;\; \delta_\omega(\alpha) = 0\}$ (see [21]).

From (3.44), it follows that

$$H^k_\omega(M) \cong \mathcal{H}^k_\omega(M). \tag{3.45}$$

Now, we will prove the announced result about the triviality of the cohomology $H^*_\omega(M)$.

Theorem 3.15 *Let M be a compact differentiable manifold and ω a closed 1-form on M, $\omega \neq 0$. Suppose that g is a Riemannian metric on M such that ω is parallel with respect to g. Then, the cohomology $H^*_\omega(M)$ is trivial.*

Proof: Since ω is parallel and non-null it follows that $\|w\| = c$, with c constant, $c > 0$. Assume, without the loss of generality, that $c = 1$. Note that if $c \neq 1$, we can consider the Riemannian metric $g' = c^2 g$ and it is clear that the module of ω with respect to g' is 1 and that ω is also parallel with respect to g'.

Under the hypothesis $c = 1$, we have that

$$\omega(U) = 1. \tag{3.46}$$

Using that ω is parallel and that U is Killing, we obtain that (see (3.39), (3.41) and [20])

$$\mathcal{L}_U = -\delta \circ e(\omega) - e(\omega) \circ \delta, \tag{3.47}$$

$$\delta \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta. \tag{3.48}$$

From (3.36), (3.37), (3.39), (3.40), (3.41), (3.46) and (3.48), we deduce the following relations:

$$d_\omega \circ i_U = -i_U \circ d_\omega + \mathcal{L}_U + Id, \qquad \delta_\omega \circ i_U = -i_U \circ \delta_\omega, \tag{3.49}$$

$$d_\omega \circ \mathcal{L}_U = \mathcal{L}_U \circ d_\omega, \qquad \delta_\omega \circ \mathcal{L}_U = \mathcal{L}_U \circ \delta_\omega, \tag{3.50}$$

where Id denotes the identity transformation.

On the other hand, (3.47) implies that

$$\begin{array}{rcl} < \mathcal{L}_U\alpha, \alpha > & = & - < \alpha, di_U\alpha + i_U d\alpha > \\ & = & - < \alpha, \mathcal{L}_U\alpha > \end{array}$$

for all $\alpha \in \Omega^k(M)$. Thus,

$$< \mathcal{L}_U\alpha, \alpha >= 0. \tag{3.51}$$

Now, if $\alpha \in \mathcal{H}^k_\omega(M)$ then, using (3.49), we have that

$$\mathcal{L}_U\alpha = -\alpha + d_\omega(i_U\alpha).$$

But, by (3.50), we deduce that $\mathcal{L}_U\alpha \in \mathcal{H}^k_\omega(M)$. Therefore (see (3.44)) we obtain that

$$\mathcal{L}_U\alpha = -\alpha.$$

Consequently, from (3.51), it follows that $\alpha = 0$.

This proves that $\mathcal{H}^k_\omega(M) = \{0\}$ which implies that $H^k_\omega(M) = \{0\}$ (see (3.45)). $\square$

Next, we will obtain some results which relate the LJ-cohomology of a l.c.s. manifold M with its de Rham cohomology and with the cohomology $H^*_\omega(M)$, ω being the Lee 1-form of M.

Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω. Suppose that (Λ, E) is the associated Jacobi structure on M and that $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. The isomorphism $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ can be extended to a mapping, which we also denote by $\flat$, from the space $\mathcal{V}^k(M)$ onto the space $\Omega^k(M)$ by putting:

$$\flat(X_1 \wedge \ldots \wedge X_k) = \flat(X_1) \wedge \ldots \wedge \flat(X_k)$$

for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$. This extension is also an isomorphism of $C^\infty(M, \mathbb{R})$-modules. In fact, we have that (see [32])

$$\#_\Lambda \alpha = (-1)^k \flat^{-1}(\alpha) \tag{3.52}$$

for all $\alpha \in \Omega^k(M)$, where $\#_\Lambda : \Omega^k(M) \to \mathcal{V}^k(M)$ is the homomorphism given by (2.20) and (2.21).

Using (2.11), (2.20), (2.21), (3.52) and the fact that $\#_\Lambda(\omega) = -E$, we obtain

$$\#_\Lambda \circ i_E = i_\omega \circ \#_\Lambda, \qquad i_E \circ \flat = -\flat \circ i_\omega. \tag{3.53}$$

Thus, from (3.32), (3.52) and (3.53), we deduce that

$$-\flat[\Lambda, P] + k\omega \wedge \flat(P) = d\flat(P) - i_E(\flat(P)) \wedge \Omega, \qquad \mathcal{L}_E\flat(P) = \flat(\mathcal{L}_E P) \tag{3.54}$$

for all $P \in \mathcal{V}^k(M)$.

Furthermore, we prove the following

Theorem 3.16 *Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω. Suppose that (Λ, E) is the associated Jacobi structure on M and that $\bar{F}^k : \Omega^k(M) \longrightarrow \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and $\bar{G}^k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \to \Omega^{k-1}(M)$ are the homomorphisms of $C^\infty(M, \mathbb{R})$-modules defined by*

$$\bar{F}^k(\alpha) = (\#_\Lambda\alpha, -\#_\Lambda(i_E\alpha)) \quad \text{and} \quad \bar{G}^k(P, Q) = (-1)^k(-\flat(Q) + i_E\flat(P))$$

for all $\alpha \in \Omega^k(M)$ and $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. Then:

(i) The mappings $\bar{F}^k$ and $\bar{G}^k$ induce an exact sequence of complexes

$$0 \longrightarrow (\Omega^*(M), d) \xrightarrow{\bar{F}} (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), -\sigma) \xrightarrow{\bar{G}} (\Omega^{*-1}(M), -d_\omega) \longrightarrow 0,$$

where d is the exterior differential, σ is the LJ-cohomology operator and d_ω is the operator given by (3.36).

(ii) This exact sequence induces a long exact cohomology sequence

$$\cdots \longrightarrow H^k_{dR}(M) \xrightarrow{\bar{F}^k_*} H^k_{LJ}(M) \xrightarrow{\bar{G}^k_*} H^{k-1}_\omega(M) \xrightarrow{L^{k-1}} H^{k+1}_{dR}(M) \longrightarrow \cdots,$$

with connecting homomorphism L^{k-1} defined by

$$L^{k-1}([\alpha]) = [\alpha \wedge \Omega]$$

for all $[\alpha] \in H^{k-1}_\omega(M)$.

Proof: It follows from (2.11), (3.7), (3.32), (3.36), (3.52), (3.53) and (3.54). □

Using Theorem 3.16, we obtain

Corollary 3.17 *Let (M, Ω) be a l.c.s. manifold of finite type with Lee 1-form ω. Suppose that the dimension of the k-th cohomology group $H^k_\omega(M)$ is finite, for all k. Then,*

$$H^k_{LJ}(M) \cong \frac{H^k_{dR}(M)}{\mathrm{Im}L^{k-2}} \oplus \ker L^{k-1},$$

where $L^r : H^r_\omega(M) \longrightarrow H^{r+2}_{dR}(M)$ is the homomorphism given by

$$L^r([\alpha]) = [\alpha \wedge \Omega]$$

for $[\alpha] \in H^r_\omega(M)$. In particular, the dimension of $H^k_{LJ}(M)$ is finite.

Remark 3.18 Theorem 3.11 follows directly from Proposition 3.14 and Corollary 3.17.

Using Theorem 3.15 and Corollary 3.17, we deduce the following results

Corollary 3.19 *Let (M, Ω) be a l.c.s. manifold of finite type with Lee 1-form ω and such that the dimension of the k-th cohomology group $H^k_\omega(M)$ is finite, for all k. If the 2-form Ω is $d_{(-\omega)}$-exact, that is, there exists a 1-form η on M such that $\Omega = d\eta - \omega \wedge \eta$, then,*

$$H^k_{LJ}(M) \cong H^k_{dR}(M) \oplus H^{k-1}_\omega(M), \qquad \textit{for all } k.$$

Corollary 3.20 *Let (M, Ω) be a compact l.c.s. manifold with Lee 1-form ω, $\omega \neq 0$. Suppose that g is a Riemannian metric on M such that ω is parallel with respect to g. Then,*

$$H^k_{LJ}(M) \cong H^k_{dR}(M), \qquad \textit{for all } k.$$

Example 3.21 Let (N, η) be a compact contact manifold and consider on the product manifold $M = N \times S^1$ the 2-form Ω defined by

$$\Omega = (pr_1)^*(d\eta) - (pr_2)^*(\theta) \wedge (pr_1)^*(\eta), \tag{3.55}$$

θ being the length element of S^1. Then, (M, Ω) is a l.c.s. manifold with Lee 1-form $\omega = (pr_2)^*(\theta)$. Furthermore, if h is a Riemannian metric on N, the 1-form ω is parallel with respect to the Riemannian metric g on M given by

$$g = (pr_1)^*(h) + \omega \otimes \omega.$$

Therefore, using Corollary 3.20, we deduce

$$H^k_{LJ}(M) \cong H^k_{dR}(M) \cong H^k_{dR}(N) \oplus H^{k-1}_{dR}(N).$$

3.6 Lichnerowicz-Jacobi cohomology of the unit sphere of a real Lie algebra

If $\mathfrak{g}$ is a real Lie algebra of finite dimension endowed with a scalar product then the unit sphere of $\mathfrak{g}$ admits a Jacobi structure (see Section 2.2). In this section, we will describe the LJ-cohomology of the sphere for the case when $\mathfrak{g}$ is the Lie algebra of a compact Lie group.

First, we will prove some results which will be useful in the sequel.

Let $(\mathfrak{g}, [\ ,\ ])$ be a real Lie algebra of dimension n and $<\ ,\ >$ a scalar product on $\mathfrak{g}$. Denote by $S^{n-1}(\mathfrak{g})$ the unit sphere in $\mathfrak{g}$.

Lemma 3.22 *If $\xi \in \mathfrak{g}$ and $\tilde{\xi} : S^{n-1}(\mathfrak{g}) \times \mathbb{R} \to \mathbb{R}$ is the real C^∞-function given by*

$$\tilde{\xi}(\eta, t) = e^t < \xi, \eta > \tag{3.56}$$

for all $(\eta, t) \in S^{n-1}(\mathfrak{g}) \times \mathbb{R}$, then

$$\frac{\partial}{\partial t}(\tilde{\xi}) = \tilde{\xi}. \tag{3.57}$$

Moreover, if $\{\xi_i\}_{i=1,\ldots n}$ is a basis of $\mathfrak{g}$ we have that the set $\{d\tilde{\xi}_i\}_{i=1,\ldots,n}$ is a global basis of the space of 1-forms on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$.

Proof: (3.57) follows directly from (3.56).

On the other hand, let $F : \mathfrak{g} - \{0\} \to S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ be the diffeomorphism defined by (2.19). Then, we deduce that

$$\tilde{\xi} \circ F = < \xi,\ \ >$$

for all $\xi \in \mathfrak{g}$, where $< \xi,\ \ >: \mathfrak{g} - \{0\} \to \mathbb{R}$ is the real function given by

$$< \xi,\ \ > (\eta) = < \xi, \eta >,$$

for all $\eta \in \mathfrak{g} - \{0\}$. This proves the second assertion of Lemma 3.22. □

Using Lemma 3.22, we obtain

Lemma 3.23 *Let P (respectively, Q) be a k-vector (respectively, a $(k-1)$-vector) on $S^{n-1}(\mathfrak{g})$. Denote by $\widetilde{(P, Q)}$ the k-vector on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ given by*

$$\widetilde{(P, Q)} = e^{-kt}(P + \frac{\partial}{\partial t} \wedge Q). \tag{3.58}$$

If $\mathcal{L}$ is the Lie derivative operator on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ then,

$$\mathcal{L}_{\frac{\partial}{\partial t}}\widetilde{(P, Q)} = -k\widetilde{(P, Q)}, \qquad \frac{\partial}{\partial t}(\widetilde{(P, Q)}(d\tilde{\xi}_1, \ldots, d\tilde{\xi}_k)) = 0 \tag{3.59}$$

for all $\xi_1, \ldots \xi_k \in \mathfrak{g}$, where $\tilde{\xi}_i$ $(i = 1, \ldots, k)$ is the real C^∞-function on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ given by (3.56).

Now, we will describe the LJ-cohomology of $S^{n-1}(\mathfrak{g})$ for the case when $\mathfrak{g}$ is the Lie algebra of a compact Lie group.

Theorem 3.24 *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G of dimension n. Suppose that $<\ ,\ >$ is a scalar product on $\mathfrak{g}$ and consider on the unit sphere $S^{n-1}(\mathfrak{g})$ the induced Jacobi structure. Then*

$$H^k_{LJ}(S^{n-1}(\mathfrak{g})) \cong H^k(\mathfrak{g}) \otimes Inv$$

for all k, where $H^*(\mathfrak{g})$ is the cohomology of $\mathfrak{g}$ relative to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and Inv is the subalgebra of $C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})$ defined by

$$Inv = \{\varphi \in C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R}) / X_f(\varphi) = 0, \ \ \forall f \in C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})\}.$$

Proof: Denote by $\overline{Ad}^*$ the action of G on $S^{n-1}(\mathfrak{g})$ given by (2.24). This action induces a representation of $\mathfrak{g}$ on the vector space $C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})$ given by

$$(\xi,\varphi) \in \mathfrak{g} \times C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R}) \to \xi_{S^{n-1}(\mathfrak{g})}(\varphi) \in C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R}),$$

$\xi_{S^{n-1}(\mathfrak{g})}$ being the infinitesimal generator, with respect to the action $\overline{Ad}^*$, associated to $\xi \in \mathfrak{g}$.

The above representation allows us to consider the differential complex

$$(C^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})), \partial)$$

and its cohomology $H^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R}))$ (see Section 3.1).

We will show that

$$H^k_{LJ}(S^{n-1}(\mathfrak{g})) \cong H^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})),$$

for all k.

Let $C^k_{HCE}(S^{n-1}(\mathfrak{g}))$ be the space of k-cochains in the H-Chevalley-Eilenberg complex of $S^{n-1}(\mathfrak{g})$. We define the homomorphism

$$\mu^k : C^k_{HCE}(S^{n-1}(\mathfrak{g})) \longrightarrow C^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R}))$$

by

$$(\mu^k(c^k))(\xi_1,\ldots,\xi_k) = c^k(<\xi_1,\ >,\ldots,<\xi_k,\ >) \tag{3.60}$$

for all $c^k \in C^k_{HCE}(S^{n-1}(\mathfrak{g}))$ and $\xi_1,\ldots,\xi_k \in \mathfrak{g}$, where $<\xi_j,\ \ >$ $(j=1,\ldots,k)$ is the real C^∞-function on $S^{n-1}(\mathfrak{g})$ given by (2.17).

Now, consider the homomorphism of $C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})$-modules

$$\Phi^k : \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g})) \to C^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R}))$$

defined by

$$\Phi^k = \mu^k \circ j^k, \tag{3.61}$$

$j^k : \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g})) \to C^k_{HCE}(S^{n-1}(\mathfrak{g}))$ being the mapping given by (3.5).

A direct computation shows that

$$
\begin{array}{rcl}
(\Phi^k(P,Q))(\xi_1,\dots,\xi_k)(\xi) & = & P(d<\xi_1,\ >,\dots,d<\xi_k,\ >)(\xi) \\
+\displaystyle\sum_{i=1}^k(-1)^{i+1}<\xi_i,\xi> Q(d<\xi_1,\ >,\dots,\widehat{d<\xi_i,\ >},\dots,d<\xi_k,\ >)(\xi) & & \\
& = & ((\widetilde{P,Q})(d\tilde{\xi}_1,\dots,d\tilde{\xi}_k))(\xi,0),
\end{array} \tag{3.62}
$$

for all $(P,Q) \in \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g}))$, $\xi_1,\dots,\xi_k \in \mathfrak{g}$ and $\xi \in S^{n-1}(\mathfrak{g})$, where $(\widetilde{P,Q})$ is the k-vector on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ defined by (3.58) and $\tilde{\xi}_i$ $(i = 1,\dots,k)$ is the function on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ given by (3.56).

Using (2.18), (2.25), (3.2), (3.4) and (3.60), we have that the mappings μ^k induce a homomorphism between the complexes $(C^*_{HCE}(S^{n-1}(\mathfrak{g})), \partial_H)$ and $(C^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})), \partial)$. Thus, the mappings Φ^k induce a homomorphism between the complexes $(\mathcal{V}^*(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{*-1}(S^{n-1}(\mathfrak{g})), \sigma)$ and $(C^*(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})), \partial)$ (see (3.6) and (3.61)).

On the other hand, if $\Phi^k(P,Q) = 0$ then, from (3.59), (3.62) and Lemma 3.22, it follows that

$$0 = (\widetilde{P,Q}) = e^{-kt}(P + \frac{\partial}{\partial t} \wedge Q).$$

Therefore, $P = 0$ and $Q = 0$.

Consequently, Φ^k is a monomorphism.

Next, we will see that Φ^k is an epimorphism.

Let $c^k : \mathfrak{g} \times \dots^{(k} \dots \times \mathfrak{g} \to C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})$ be a $C^\infty(S^{n-1}(\mathfrak{g}),\mathbb{R})$-valued k-cochain.

We define a k-vector R on $S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ characterized by the condition

$$R(d\tilde{\xi}_1,\dots d\tilde{\xi}_k)(\xi,t) = e^{kt}(c^k(\xi_1,\dots,\xi_k)(\xi)) \tag{3.63}$$

for all $\xi_1,\dots,\xi_k \in \mathfrak{g}$ and $(\xi,t) \in S^{n-1}(\mathfrak{g}) \times \mathbb{R}$.

From Lemma 3.22, we deduce that R is well-defined and, using (3.57) and (3.63), we have that

$$\mathcal{L}_{\frac{\partial}{\partial t}} R = 0.$$

This implies that

$$R = P + \frac{\partial}{\partial t} \wedge Q, \tag{3.64}$$

with $(P,Q) \in \mathcal{V}^k(S^{n-1}(\mathfrak{g})) \oplus \mathcal{V}^{k-1}(S^{n-1}(\mathfrak{g}))$.

Moreover, from (3.58), (3.62), (3.63) and (3.64), it follows that

$$\Phi^k(P,Q) = c^k.$$

Thus, Φ^k is an epimorphism.

Using the above facts, we conclude that

$$H^k_{LJ}(S^{n-1}(\mathfrak{g})) \cong H^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})),$$

for all k.

Now, if we apply a general result of Ginzburg and Weinstein (see Theorem 3.5 of [19]; see also [13]), we obtain that

$$H^k(\mathfrak{g}; C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})) \cong H^k(\mathfrak{g}) \otimes \overline{Inv}$$

where $\overline{Inv}$ is the algebra of G-invariant functions on $S^{n-1}(\mathfrak{g})$ with respect to the action $\overline{Ad}^*$.

Finally, from (2.25) and since the characteristic foliation of $S^{n-1}(\mathfrak{g})$ is generated by the set of hamiltonian vector fields

$$\{X_{<\xi,\ >}/\xi \in \mathfrak{g}\},$$

we deduce that $\overline{Inv} = Inv$. □

It is well-known that if $\mathfrak{g}$ is the Lie algebra of a compact semisimple Lie group then $H^2(\mathfrak{g}) = \{0\}$. Therefore, using Theorem 3.24, we have

Corollary 3.25 *Let $\mathfrak{g}$ be the Lie algebra of a compact semisimple Lie group G of dimension n. Suppose that $<\ ,\ >$ is a scalar product on $\mathfrak{g}$ and consider on the unit sphere $S^{n-1}(\mathfrak{g})$ the induced Jacobi structure. Then*

$$H^2_{LJ}(S^{n-1}(\mathfrak{g})) = \{0\}.$$

3.7 Table I

The following table summarizes the main results obtained in Sections 3.3, 3.4, 3.5 and 3.6 about the LJ-cohomology of the different types of Jacobi manifolds.

TYPE	LJ-COHOMOLOGY	REMARKS
(M,Ω) symplectic of finite type	$H^k_{LJ}(M)\cong \dfrac{H^k_{dR}(M)}{\mathrm{Im}L^{k-2}}\oplus \ker L^{k-1}$	$L^r: H^r_{dR}(M)\longrightarrow H^{r+2}_{dR}(M)$ $[\alpha]\mapsto[\alpha\wedge\Omega]$
M exact symplectic of finite type	$H^k_{LJ}(M)\cong H^k_{dR}(M)\oplus H^{k-1}_{dR}(M)$	$\dim H^k_{LJ}(M)=b_k(M)+b_{k-1}(M)$
M^{2m} Lefschetz symplectic of finite type	$H^k_{LJ}(M)\cong \dfrac{H^k_{dR}(M)}{\mathrm{Im}L^{k-2}},\quad k\le m$ $H^k_{LJ}(M)\cong \ker L^{k-1},\quad k\ge m+1$	$\dim H^k_{LJ}(M)=b_k(M)-b_{k-2}(M)$ $k\le m$ $\dim H^k_{LJ}(M)=b_{k-1}(M)-b_{k+1}(M)$ $k\ge m+1$
$M=\Gamma\backslash G$ compact symplectic nilmanifold	$H^k_{LJ}(M)\cong \dfrac{H^k(\mathfrak{g})}{\mathrm{Im}(L_{\mathfrak{g}})^{k-2}}\oplus \ker(L_{\mathfrak{g}})^{k-1}$	$\mathfrak{g}$ Lie algebra of G $\tilde{\Omega}_{\mathfrak{g}}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ induced symplectic form $(L_{\mathfrak{g}})^r: H^r(\mathfrak{g})\to H^{r+2}(\mathfrak{g})$ $[\alpha]\mapsto[\alpha\wedge\tilde{\Omega}_{\mathfrak{g}}]$
Dual of a real Lie algebra $\mathfrak{g}$ of finite dimension	$H^k_{LJ}(\mathfrak{g}^*)\cong H^k_{LP}(\mathfrak{g}^*)\oplus H^{k-1}_{LP}(\mathfrak{g}^*)$	
Dual of the Lie algebra $\mathfrak{g}$ of a compact Lie group	$H^k_{LJ}(\mathfrak{g}^*)\cong (H^k(\mathfrak{g})\otimes Inv)\oplus$ $(H^{k-1}(\mathfrak{g})\otimes Inv)$	$Inv\equiv$ subalgebra of the Casimir functions of $\mathfrak{g}^*$
M contact	$H^k_{LJ}(M)\cong H^k_{dR}(M)\oplus H^{k-1}_{dR}(M)$	
(M,Ω) g.c.s. of finite type with Lee 1-form $\omega=df$	$H^k_{LJ}(M)\cong \dfrac{H^k_{dR}(M)}{\mathrm{Im}\bar{L}^{k-2}}\oplus \ker \bar{L}^{k-1}$	$\bar{L}^r: H^r_{dR}(M)\to H^{r+2}_{dR}(M)$ $[\alpha]\mapsto[e^{-f}\alpha\wedge\Omega]$
(M,Ω) l.c.s. of finite type with Lee 1-form ω and $\dim H^*_\omega(M)<\infty$	$H^k_{LJ}(M)\cong \dfrac{H^k_{dR}(M)}{\mathrm{Im}L^{k-2}}\oplus \ker L^{k-1}$	$L^r: H^r_\omega(M)\to H^{r+2}_{dR}(M)$ $[\alpha]\mapsto[\alpha\wedge\Omega]$
M compact l.c.s. with Lee 1-form ω ω parallel with respect to a Riemannian metric	$H^k_{LJ}(M)\cong H^k_{dR}(M)$	$\dim H^k_{LJ}(M)=b_k(M)$
Unit sphere $S^{n-1}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of a compact Lie group ($\dim\mathfrak{g}=n$)	$H^k_{LJ}(S^{n-1}(\mathfrak{g}))\cong H^k(\mathfrak{g})\otimes Inv$	$Inv\equiv$ subalgebra of the constant functions on the leaves of the characteristic foliation

Table I: LJ-cohomology

4 Lichnerowicz-Jacobi homology of a Jacobi manifold

4.1 H-Chevalley-Eilenberg homology and Lichnerowicz-Jacobi homology of a Jacobi manifold

In a similar way that for the cohomology, firstly we recall the definition of the homology of a Lie algebra $\mathcal{A}$ with coefficients in an $\mathcal{A}$-module (see, for instance, [7]).

Let $(\mathcal{A}, [\ ,\ ])$ be a real Lie algebra (not necessarily finite dimensional) and $\mathcal{M}$ a real vector space endowed with a $\mathbb{R}$-bilinear multiplication

$$\mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M}, \qquad (a, m) \mapsto a.m$$

compatible with the bracket $[\ ,\ ]$, i.e., such that (3.1) holds.

An *$\mathcal{M}$-valued k-chain* is an element of the vector space $C_k(\mathcal{A};\mathcal{M}) = \mathcal{M} \otimes \Lambda^k\mathcal{A}$, where $\Lambda^*\mathcal{A}$ is the exterior algebra of $\mathcal{A}$. We can consider the linear operator $\delta_k : C_k(\mathcal{A};\mathcal{M}) \longrightarrow C_{k-1}(\mathcal{A};\mathcal{M})$ characterized by

$$\begin{aligned} \delta_k(m \otimes (a_1 \wedge \ldots \wedge a_k)) &= \sum_{1\le i\le k} (-1)^i a_i.m \otimes (a_1 \wedge \ldots \wedge \widehat{a_i} \wedge \ldots \wedge a_k) + \\ & \sum_{1\le i<j\le k} (-1)^{i+j} m \otimes ([a_i, a_j] \wedge a_1 \wedge \ldots \wedge \widehat{a_i} \wedge \ldots \wedge \widehat{a_j} \wedge \ldots \wedge a_k), \end{aligned} \tag{4.1}$$

which satisfies $\delta_{k-1} \circ \delta_k = 0$, for all k. Then, we have the corresponding homology spaces

$$H_k(\mathcal{A};\mathcal{M}) = \frac{\ker\{\delta_k : C_k(\mathcal{A};\mathcal{M}) \to C_{k-1}(\mathcal{A};\mathcal{M})\}}{\mathrm{Im}\{\delta_{k+1} : C_{k+1}(\mathcal{A};\mathcal{M}) \to C_k(\mathcal{A};\mathcal{M})\}}.$$

This homology is said to be the *homology of the Lie algebra $\mathcal{A}$ with coefficients in $\mathcal{M}$ or relative to the given representation of $\mathcal{A}$ on $\mathcal{M}$.*

Now, let (M, Λ, E) be a Jacobi manifold and $\{\ ,\ \}$ the associated Jacobi bracket. We consider the homology of the Lie algebra $(C^\infty(M,\mathbb{R}), \{\ ,\ \})$ relative to the representation defined by the hamiltonian vector fields as in (3.3). This homology is called the *H-Chevalley-Eilenberg homology associated to M.*

We denote by $C_k^{HCE}(M)$ the space of the k-chains in the H-Chevalley-Eilenberg complex, by δ_H the homology operator and by $H_k^{HCE}(M)$ the k-th homology group. Then, if $f \otimes (f_1 \wedge \ldots \wedge f_k) \in C_k^{HCE}(M) = C^\infty(M,\mathbb{R}) \otimes (\Lambda^k(C^\infty(M,\mathbb{R})))$,

$$\begin{aligned} \delta_H(f \otimes (f_1 \wedge \ldots \wedge f_k)) &= \sum_{1\le i\le k} (-1)^i X_{f_i}(f) \otimes (f_1 \wedge \ldots \wedge \widehat{f_i} \wedge \ldots \wedge f_k) + \\ & \sum_{1\le i<j\le k} (-1)^{i+j} f \otimes (\{f_i, f_j\} \wedge f_1 \wedge \ldots \wedge \widehat{f_i} \wedge \ldots \wedge \widehat{f_j} \wedge \ldots \wedge f_k). \end{aligned} \tag{4.2}$$

On the other hand, the skew-symmetric k-multilinear mapping $\tilde{\pi}_k : C^\infty(M,\mathbb{R}) \times \ldots^{(k} \ldots \times C^\infty(M,\mathbb{R}) \to \Omega^k(M) \otimes \Omega^{k-1}(M)$ defined by

$$\tilde{\pi}_k(f_1, \ldots, f_k) = (df_1 \wedge \ldots \wedge df_k, \sum_{i=1}^{k} (-1)^{i+k} f_i df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k)$$

induces a linear mapping $\pi_k : C_k^{HCE}(M) \to \Omega^k(M) \oplus \Omega^{k-1}(M)$ characterized by

$$\pi_k(f \otimes (f_1 \wedge \ldots \wedge f_k)) = (f df_1 \wedge \ldots \wedge df_k, \sum_{i=1}^{k} (-1)^{i+k} f f_i df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k) \tag{4.3}$$

for all $f \otimes (f_1 \wedge \ldots \wedge f_k) \in C_k^{HCE}(M)$.

A direct computation, using (2.2), (2.22), (4.2) and (4.3), shows that

$$\delta \circ \pi_k = \pi_{k-1} \circ \delta_H, \tag{4.4}$$

where $\delta : \Omega^r(M) \oplus \Omega^{r-1}(M) \longrightarrow \Omega^{r-1}(M) \oplus \Omega^{r-2}(M)$ is the operator given by

$$\begin{array}{rcl} \delta(\alpha, \beta) & = & (i(\Lambda)d\alpha - di(\Lambda)\alpha + r i_E \alpha + (-1)^r \mathcal{L}_E \beta, \\ & & i(\Lambda)d\beta - di(\Lambda)\beta + (r-1) i_E \beta + (-1)^r i(\Lambda)\alpha), \end{array} \tag{4.5}$$

$i(\Lambda)$ being the contraction by Λ.

Since the mappings π_k are locally surjective, from (4.4), it follows that

$$\delta^2 = 0. \tag{4.6}$$

This fact allows us to consider the differential complex

$$\cdots \longrightarrow \Omega^{k+1}(M) \oplus \Omega^k(M) \stackrel{\delta}{\longrightarrow} \Omega^k(M) \oplus \Omega^{k-1}(M) \stackrel{\delta}{\longrightarrow} \Omega^{k-1}(M) \oplus \Omega^{k-2}(M) \longrightarrow \cdots$$

whose homology is called the *Lichnerowicz-Jacobi homology* (LJ-homology) of M and denoted by $H_*^{LJ}(M, \Lambda, E)$ or simply by $H_*^{LJ}(M)$ if there is not danger of confusion.

Remark 4.1 Let $\Omega_B^k(M)$ be the space of the basic k-forms with respect to E, that is, $\alpha \in \Omega_B^k(M)$ if and only if

$$i_E \alpha = 0, \qquad \mathcal{L}_E \alpha = 0.$$

Denote by $\bar{\delta}$ the homology operator of the subcomplex of the LJ-complex which consists of the pairs $(0, \alpha)$, α being a basic form with respect to E. Under the canonical identification $\{0\} \oplus \Omega_B^k(M) \cong \Omega_B^k(M)$ one has that

$$\bar{\delta}\alpha = i(\Lambda)d\alpha - di(\Lambda)\alpha, \qquad \text{for all } \alpha \in \Omega_B^k(M).$$

The homology of the complex $(\Omega_B^*(M), \bar{\delta})$ was studied in [10] and [11] and it was called the *canonical homology of the Jacobi manifold* M. This name is justified by the fact that if M is a Poisson manifold ($E = 0$), then the homology of the complex $(\Omega_B^*(M), \bar{\delta})$ is just the canonical homology introduced by Brylinski [5] (see also [29]). Note that $d\bar{\delta} + \bar{\delta}d = 0$ and thus one can consider a double complex and the two spectral sequences associated with it. The degeneration of these spectral sequences at the first term and other related aspects were discussed in [5, 16, 17, 23, 44] (for the case of a Poisson manifold) and in [10, 11] (for the case of a Jacobi manifold).

4.2 Modular class of a Jacobi manifold and duality between the Lichnerowicz-Jacobi cohomology and the Lichnerowicz-Jacobi homology

In this section, we will show that the LJ-homology of a Jacobi manifold (M, Λ, E) of dimension n is just the homology of the Lie algebroid $(T^*M\times\mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ with respect to a certain flat $(T^*M\times\mathbb{R})$-connection on $\Lambda^{n+1}(T^*M\times\mathbb{R})$. This last homology was introduced by Vaisman in [53]. Moreover, in this paper, Vaisman also introduced the definition of the modular class of an orientable Jacobi manifold and he proved that if such a class is zero then there is a duality between the LJ-homology and the LJ-cohomology.

Firstly, we will recall several results of [59] (see also [14, 28, 42]).

Let $(K, [\![\ ,\ ]\!], \varrho)$ be a Lie algebroid over M.

Denote by $[\![\ ,\ ]\!]_{\mathcal{A}} : \Gamma(\Lambda^{r_1}K)\times\Gamma(\Lambda^{r_2}K) \to \Gamma(\Lambda^{r_1+r_2-1}K)$ the bracket characterized by the relations

$$[\![f,g]\!]_{\mathcal{A}} = 0, \qquad [\![X,f]\!]_{\mathcal{A}} = \varrho(X)(f), \qquad [\![X,Y]\!]_{\mathcal{A}} = [\![X,Y]\!],$$
$$[\![U, V\wedge W]\!]_{\mathcal{A}} = [\![U,V]\!]_{\mathcal{A}}\wedge W + (-1)^{(r_1+1)s_1} V\wedge [\![U,W]\!]_{\mathcal{A}},$$

for all $f, g \in C^\infty(M,\mathbb{R})$, $X, Y \in \Gamma(K)$, $U\in\Gamma(\Lambda^{r_1}K)$, $V\in\Gamma(\Lambda^{s_1}K)$ and $W\in\Gamma(\Lambda^{s_2}K)$. Then, if n is the rank of K, it follows that $(\mathcal{A} = \bigoplus_{0\le r\le n}\Gamma(\Lambda^r K), [\![\ ,\ ]\!]_{\mathcal{A}})$ is a *Gerstenhaber algebra* (see [14, 28, 42, 59]).

On the other hand, a *K-connection on a vector bundle $L\to M$* is a $\mathbb{R}$-bilinear mapping

$$\nabla : \Gamma(K)\times\Gamma(L)\to\Gamma(L), \qquad (X,s)\mapsto\nabla_X s$$

such that

$$\nabla_{fX}s = f\nabla_X s, \qquad \nabla_X fs = f\nabla_X s + \varrho(X)(f)s, \qquad \text{for all } f\in C^\infty(M,\mathbb{R}).$$

The curvature R of a K-connection ∇ may be defined as for the usual connections. ∇ is said to be *flat* if R vanishes.

Any K-connection on $\Lambda^n K\to M$ defines a differential operator $D:\Gamma(\Lambda^r K)\to\Gamma(\Lambda^{r-1}K)$ locally given by

$$D(i(\omega)\Phi) = (-1)^{n-k+1}(i(\tilde{\partial}^{n-r}\omega)\Phi + \sum_{i=1}^{n}\alpha^h\wedge i(w)\nabla_{X_h}\Phi), \tag{4.7}$$

where $\Phi\in\Gamma(\Lambda^n K)$, $\omega\in\Gamma(\Lambda^{n-r}K^*)$, $\{X_h\}$ is a local basis of $\Gamma(K)$ and $\{\alpha^h\}$ is the dual basis of $\Gamma(K^*)$. The operator D generates the Gerstenhaber algebra $(\mathcal{A}, [\![\ ,\ ]\!]_{\mathcal{A}})$, that is, for all $U_1\in\Gamma(\Lambda^{r_1}K)$ and $U_2\in\Gamma(\Lambda^{r_2}K)$

$$[\![U_1,U_2]\!]_{\mathcal{A}} = (-1)^{r_1}(D(U_1\wedge U_2) - DU_1\wedge U_2 - (-1)^{r_1}U_1\wedge DU_2).$$

Moreover, the connection ∇ can be recovered from the operator D. More precisely, we have that

$$\nabla_X \Phi = -X \wedge D\Phi, \tag{4.8}$$

for all $X \in \Gamma(K)$ and $\Phi \in \Gamma(\Lambda^n K)$.

In fact, (4.7) and (4.8) define a one-to-one correspondence between K-connections on $\Lambda^n K$ and linear operators D generating the Gerstenhaber algebra $(\mathcal{A}, [\![\ ,\ ]\!]_{\mathcal{A}})$. Under this correspondence, a flat K-connection ∇ corresponds to a operator D of square zero. Thus, a flat K-connection ∇ induces a homology operator. The corresponding homology $H_*(K,\nabla)$ is the *homology of the Lie algebroid K with respect to the flat K-connection ∇.* For two flat K-connections ∇ and ∇' such that their generating operators, D and D', satisfy $D - D' = i(\tilde{\partial} f)$, with $f \in C^\infty(M,\mathbb{R})$, one has that $H_r(K,\nabla) \cong H_r(K,\nabla')$, for all r. Furthermore, if $\nu \in \Gamma(\Lambda^n K)$ is such that $\nu(x) \neq 0$, for all $x \in M$, and $\nabla \nu = 0$ then it is possible to define a duality between the homology $H_*(K,\nabla)$ and the cohomology of the Lie algebroid K with trivial coefficients $H^*(K)$. More precisely, the mapping $\star : \Gamma(\Lambda^r K^*) \to \Gamma(\Lambda^{n-r} K)$ given by

$$\star\xi = i(\xi)\nu,$$

induces an isomorphism between the cohomology group $H^r(K)$ and the homology group $H_{n-r}(K,\nabla)$ (for more details, see [59]).

Now, let (M,Λ,E) be a Jacobi manifold of dimension n and $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ its associated Lie algebroid (see Section 2.4).

The space $\Gamma(\Lambda^r(T^*M \times \mathbb{R}))$ can be identified with $\Omega^r(M) \oplus \Omega^{r-1}(M)$ in such a way that the exterior product of a section (α,β) of $\Lambda^r(T^*M \times \mathbb{R}) \to M$ with a section (α',β') of $\Lambda^{r'}(T^*M \times \mathbb{R}) \to M$ is given by

$$(\alpha,\beta) \wedge (\alpha',\beta') = (\alpha \wedge \alpha', \alpha \wedge \beta' + (-1)^{r'} \beta \wedge \alpha'). \tag{4.9}$$

On the other hand, under the identification of $\Gamma(\Lambda^k(T^*M \times \mathbb{R})^*)$ with $\mathcal{V}^k(M) \otimes \mathcal{V}^{k-1}(M)$ (see Section 3.1) the interior product of a section (α,β) of $\Lambda^r(T^*M \times \mathbb{R}) \to M$ by a section (P,Q) of $\Lambda^k(T^*M \times \mathbb{R})^* \to M$ is given by

$$\begin{aligned} &\iota(P,Q)(\alpha,\beta) = (i(P)\alpha + (-1)^{r-1} i(Q)\beta, i(P)\beta) && \text{if } k \leq r \\ &\iota(P,Q)(\alpha,\beta) = 0 && \text{if } k > r. \end{aligned}$$

In particular,

$$\iota(X,f)(\alpha,\beta) = (i(X)\alpha + (-1)^{r-1} f\beta, i(X)\beta), \tag{4.10}$$

for all $(X,f) \in \Gamma(\Lambda^1(T^*M \times \mathbb{R})^*) \cong \mathfrak{X}(M) \times C^\infty(M,\mathbb{R})$.

The Jacobi structure (Λ,E) allows us to introduce a flat $(T^*M \times \mathbb{R})$-connection on $\Lambda^{n+1}(T^*M \times \mathbb{R})$ defined by (see [53])

$$\nabla_{(\alpha,f)}(0,\Phi) = (0, f d i_E \Phi + \alpha \wedge (d i(\Lambda)\Phi - n i_E \Phi)), \tag{4.11}$$

for all $(\alpha, f) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ and $\Phi \in \Omega^n(M)$. Then, if $\delta : \Omega^r(M) \oplus \Omega^{r-1}(M) \to \Omega^{r-1}(M) \oplus \Omega^{r-2}(M)$ is the LJ-homology operator (see (4.5)) and D is the homology operator associated with ∇, we have that $D = \delta$ (see relation (2.10) in [53]). Therefore,

Proposition 4.2 *[53] Let (M, Λ, E) be a Jacobi manifold of dimension n. Then the LJ-homology $H_*^{LJ}(M)$ is the homology $H_*(T^*M \times \mathbb{R}, \nabla)$ of the Lie algebroid $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ with respect to the flat $(T^*M \times \mathbb{R})$-connection on $\Lambda^{n+1}(T^*M \times \mathbb{R})$ defined by (4.11).*

Next, assume that M is orientable and let ν be a volume form. The volume form ν induces a flat $(T^*M \times \mathbb{R})$-connection ∇_0 on $\Lambda^{n+1}(T^*M \times \mathbb{R})$ by putting

$$(\nabla_0)_{(\alpha,f)}(0,\nu) = (0,0), \qquad \text{for all } (\alpha, f) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}).$$

Then, using (4.5), (4.7), (4.8) and (4.9), we have that for all $(\alpha, f) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$,

$$\begin{array}{rcl} \nabla_{(\alpha,f)}(0,\nu) - (\nabla_0)_{(\alpha,f)}(0,\nu) & = & -(\alpha, f) \wedge \delta(0,\nu) \\ & = & (0, (f div_\nu E - n\alpha(E))\nu + \alpha \wedge di(\Lambda)\nu), \end{array} \tag{4.12}$$

where $div_\nu E$ is the divergence of the vector field E with respect to ν, that is,

$$\mathcal{L}_E \nu = (div_\nu E)\nu.$$

Now, let $\mathcal{X}^\nu_{(\Lambda,E)}$ be the vector field characterized by the relation

$$\mathcal{L}_{\#_\Lambda(df)}\nu = \mathcal{X}^\nu_{(\Lambda,E)}(f)\nu, \qquad \text{for all } f \in C^\infty(M,\mathbb{R}). \tag{4.13}$$

Using (4.13) and the fact that

$$i_{\#_\Lambda(\alpha)}\nu = -\alpha \wedge i(\Lambda)\nu, \tag{4.14}$$

for all $\alpha \in \Omega^1(M)$, it follows that

$$\begin{array}{rcl} \alpha(\mathcal{X}^\nu_{(\Lambda,E)})\nu & = & \mathcal{L}_{\#_\Lambda(\alpha)}\nu + d\alpha \wedge i(\Lambda)\nu \\ & = & \alpha \wedge di(\Lambda)\nu. \end{array} \tag{4.15}$$

Therefore,

$$\nabla_{(\alpha,f)}(0,\nu) - (\nabla_0)_{(\alpha,f)}(0,\nu) = (0, (f div_\nu E + \alpha(\mathcal{X}^\nu_{(\Lambda,E)} - nE))\nu). \tag{4.16}$$

Denote by D_0 the corresponding homology operator associated with ∇_0. From (4.7) and (4.16), we deduce that

$$D - D_0 = \iota(\mathcal{X}^\nu_{(\Lambda,E)} - nE, div_\nu E). \tag{4.17}$$

The pair

$$\mathcal{M}^\nu_{(\Lambda,E)} = (\mathcal{X}^\nu_{(\Lambda,E)} - nE, div_\nu E) \in \mathfrak{X}(M) \times C^\infty(M,\mathbb{R}) \tag{4.18}$$

defines a 1-cocycle in the LJ-complex of M, that is, $\sigma(\mathcal{M}^{\nu}_{(\Lambda,E)}) = (0,0)$, where σ is the LJ-cohomology operator. Moreover, the corresponding cohomology class $\mathcal{M}_{(\Lambda,E)} \in H^1_{LJ}(M)$ does not depend of the volume form ν (see [53]).

This cohomology class $\mathcal{M}_{(\Lambda,E)}$ is called the *Jacobi modular class of* M (see [53]). The manifold M is said to be *a unimodular Jacobi manifold* if the Jacobi modular class $\mathcal{M}_{(\Lambda,E)}$ is zero. In such a case, using (4.17), Proposition 4.2 and the results of [59] described above, we conclude the following

Theorem 4.3 *[53] If (M,Λ,E) is a unimodular Jacobi manifold of dimension n then*

$$H^{LJ}_r(M) \cong H^{n+1-r}_{LJ}(M)$$

for all $r \in \{0,\ldots,n+1\}$.

Remark 4.4 Let (M,Λ) be an orientable Poisson manifold.

(i) Suppose that ν is a volume form on M. The *modular vector field* of M with respect to ν is the vector field $\mathcal{X}^{\nu}_{\Lambda}$ characterized by

$$\mathcal{L}_{\#_\Lambda(df)}\nu = \mathcal{X}^{\nu}_{\Lambda}(f)\nu, \qquad \text{for all } f \in C^\infty(M,\mathbb{R}). \tag{4.19}$$

If $\bar{\sigma}$ denotes the LP-cohomology operator (see (3.15)) we have that $\bar{\sigma}(\mathcal{X}^{\nu}_{\Lambda}) = 0$. Thus, $\mathcal{X}^{\nu}_{\Lambda}$ defines a cohomology class $\mathcal{M}_\Lambda \in H^1_{LP}(M)$. This class does not depend of the volume form ν and it is called the *Poisson modular class* of M. If $\mathcal{M}_\Lambda$ is zero then M is said to be a *unimodular Poisson manifold* (for more details, we refer to [55, 56]).

(ii) Let $(Id^k,0) : H^k_{LP}(M) \to H^k_{LJ}(M)$ be the canonical homomorphism given by

$$(Id^k,0)([P]) = [(P,0)], \qquad \text{for } [P] \in H^k_{LP}(M).$$

Since the 0-cochains in the LP-complex and in the LJ-complex are the C^∞ real-valued functions on M, we deduce that $(Id^1,0) : H^1_{LP}(M) \to H^1_{LJ}(M)$ is a monomorphism.

On the other hand, from (4.13), (4.18) and (4.19), it follows that

$$(Id^1,0)\mathcal{M}_\Lambda = \mathcal{M}_{(\Lambda,0)},$$

where $\mathcal{M}_{(\Lambda,0)}$ is the Jacobi modular class of M. Therefore, we conclude that (M,Λ) is a unimodular Poisson manifold if and only if $(M,\Lambda,0)$ is a unimodular Jacobi manifold.

Remark 4.5 Let (M,Λ,E) be an orientable Jacobi manifold and $(M\times\mathbb{R},\tilde{\Lambda})$ the poissonization of M.

(i) Suppose that ν is a volume form on M and consider in $M \times \mathbb{R}$ the volume form

$$\tilde{\nu} = e^{(n+1)t} \nu \wedge dt,$$

where n is the dimension of M and t is the usual coordinate on $\mathbb{R}$. Using the results of Vaisman (see relations (3.13) and (3.14) in [53]) we deduce that the modular vector field $\mathcal{X}_{\tilde{\Lambda}}^{\tilde{\nu}}$ of $(M \times \mathbb{R}, \tilde{\Lambda})$ with respect to $\tilde{\nu}$ is

$$\mathcal{X}_{\tilde{\Lambda}}^{\tilde{\nu}} = e^{-t}(\mathcal{X}_{(\Lambda,E)}^{\nu} - nE + (div_{\nu} E)\frac{\partial}{\partial t}).$$

Thus, from (4.18), we conclude that $\mathcal{X}_{\tilde{\Lambda}}^{\tilde{\nu}}$ is zero if and only if $\mathcal{M}_{(\Lambda,E)}^{\nu}$ is zero.

(ii) Using again the results of Vaisman [53], we have that if (M, Λ, E) is a unimodular Jacobi manifold then $(M \times \mathbb{R}, \tilde{\Lambda})$ is a unimodular Poisson manifold. However, in general, the converse does not hold. In fact, the poissonization of a contact manifold M is unimodular (see Remark 2.2 and Section 4.4.1) and the LJ-cohomology and the LJ-homology of M are not dual one each other (see Theorems 3.9 and 4.17).

4.3 Lichnerowicz-Jacobi homology and conformal changes of Jacobi structures

In this section, we will show that the LJ-homology is also invariant under conformal changes.

Suppose that $(K, [\![\ ,\ ]\!], \varrho)$ (respectively, $(K', [\![\ ,\ ]\!]', \varrho'))$ is a Lie algebroid over M of rank n and that $\phi : K \to K'$ is an isomorphism of Lie algebroids (see Section 3.2). Denote by $\phi_1 : \Gamma(K) \to \Gamma(K')$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules induced by ϕ. This isomorphism can be extended to an isomorphism $\phi_r : \Gamma(\Lambda^r K) \to \Gamma(\Lambda^r K')$ by putting

$$\phi_r(X_1 \wedge \ldots \wedge X_r) = \phi_1(X_1) \wedge \ldots \wedge \phi_1(X_r), \tag{4.20}$$

for all $X_1, \ldots, X_r \in \Gamma(K)$.

Moreover, we have

Proposition 4.6 *Let ∇ (respectively, ∇') be a flat K-connection (respectively, a K'-connection) on $\Lambda^n K \to M$ (respectively, $\Lambda^n K' \to M$) such that ∇ and ∇' are ϕ-related, that is,*

$$\phi_n(\nabla_X \Phi) = \nabla'_{\phi_1(X)} \phi_n(\Phi), \tag{4.21}$$

for all $X \in \Gamma(K)$ and $\Phi \in \Gamma(\Lambda^n K)$. Then,

$$\phi_r \circ D = D' \circ \phi_{r+1}, \tag{4.22}$$

where D (respectively, D') is the homology operator associated with ∇ (respectively, ∇'). Thus, the Lie algebroid homologies $H_*(K, \nabla)$ and $H_*(K', \nabla')$ are isomorphic.

Proof: Denote by $\phi^r : \Gamma(\Lambda^r(K')^*) \to \Gamma(\Lambda^r K^*)$ the isomorphism of $C^\infty(M,\mathbb{R})$-modules given by (3.9). A direct computation proves that

$$\phi_r(i(\phi^{n-r}\omega')\Phi) = i(\omega')(\phi_n\Phi), \tag{4.23}$$

for all $\omega' \in \Gamma(\Lambda^{n-r}(K')^*)$ and $\Phi \in \Gamma(\Lambda^n K)$.

Therefore, using (3.10), (4.7), (4.21) and (4.23), we deduce (4.22). $\square$

Now, we have the following

Theorem 4.7 *Let (M, Λ, E) be a Jacobi manifold and (Λ_a, E_a) a conformal change of the Jacobi structure (Λ, E). Then*

$$H_k^{LJ}(M, \Lambda, E) \cong H_k^{LJ}(M, \Lambda_a, E_a),$$

for all k.

Proof: We consider the isomorphism ϕ given by (3.12) between the Lie algebroids $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, (\#_\Lambda, E))$ and $(T^*M \times \mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda_a,E_a)}, (\#_{\Lambda_a}, E_a))$.

From (3.13), (4.9) and (4.20), we obtain that

$$\phi_{n+1}(0, \Phi) = (0, \frac{1}{a^{n+1}}\Phi), \tag{4.24}$$

for $\Phi \in \Omega^n(M)$, where n is the dimension of M.

On the other hand, if ∇ (respectively, ∇^a) is the flat $(T^*M \times \mathbb{R})$-connection on $\Lambda^{n+1}(T^*M \times \mathbb{R}) \to M$ defined by (4.11) associated with the Jacobi structure (Λ, E) (respectively, (Λ_a, E_a)) then, using (3.13), (4.11), (4.14) and (4.24), we prove that

$$\phi_{n+1}(\nabla_{(\alpha,f)}(0, \Phi)) = \nabla^a_{\phi_1(\alpha,f)}\phi_{n+1}(0, \Phi),$$

for all $(\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ and $\Phi \in \Omega^n(M)$.

Consequently, the result follows from Propositions 4.2 and 4.6. $\square$

Finally, using Theorem 4.7, we deduce the result announced at the begining of this section

Corollary 4.8 *The LJ-homology is invariant under conformal changes of the Jacobi structure.*

4.4 Lichnerowicz-Jacobi homology of a Poisson manifold

Let (M, Λ) be a Poisson manifold and δ (respectively, $\bar{\delta}$) the operator of the LJ-homology (respectively, of the canonical homology) associated with M. Then,

$$
\begin{aligned}
\bar{\delta}\alpha &= i(\Lambda)d\alpha - di(\Lambda)\alpha \\
\delta(\alpha, \beta) &= (\bar{\delta}\alpha, \bar{\delta}\beta + (-1)^k i(\Lambda)\alpha)
\end{aligned}
\tag{4.25}
$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$ (see (4.5) and Remark 4.1).

Using (4.25), we obtain the following relation between the LJ-homology $H_*^{LJ}(M)$ and the canonical homology $H_*^{can}(M)$.

Theorem 4.9 *Let (M, Λ) be a Poisson manifold. Suppose that $(0, Id_k) : \Omega^{k-1}(M) \to \Omega^k(M) \oplus \Omega^{k-1}(M)$ and $(\pi_1)_k : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \Omega^k(M)$ are the homomorphisms of $C^\infty(M, \mathbb{R})$-modules given by*

$$
(0, Id_k)(\beta) = (0, \beta), \qquad (\pi_1)_k(\alpha, \beta) = \alpha
$$

for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. Then:

(i) The mappings $(0, Id_k)$ and $(\pi_1)_k$ define an exact sequence of complexes

$$
0 \longrightarrow (\Omega^{*-1}(M), \bar{\delta}) \stackrel{(0,Id)}{\longrightarrow} (\Omega^*(M) \oplus \Omega^{*-1}(M), \delta) \stackrel{\pi_1}{\longrightarrow} (\Omega^*(M), \bar{\delta}) \longrightarrow 0
$$

(ii) The above exact sequence induces a long exact homology sequence

$$
\cdots \longrightarrow H_{k-1}^{can}(M) \stackrel{(0,Id_k)_*}{\longrightarrow} H_k^{LJ}(M) \stackrel{((\pi_1)_k)_*}{\longrightarrow} H_k^{can}(M) \stackrel{\Lambda_k}{\longrightarrow} H_{k-2}^{can}(M) \longrightarrow \cdots
$$

where the connecting homomorphism Λ_k is defined by

$$
\Lambda_k[\alpha] = (-1)^k [i(\Lambda)(\alpha)] \tag{4.26}
$$

for all $[\alpha] \in H_k^{can}(M)$.

From Theorem 4.9, it follows that

Corollary 4.10 *Let M be a Poisson manifold such that its groups of canonical homology have finite dimension. Then, the LJ-homology groups have also finite dimension and*

$$
H_k^{LJ}(M) \cong \frac{H_{k-1}^{can}(M)}{\mathrm{Im}\Lambda_{k+1}} \oplus \ker \Lambda_k,
$$

where $\Lambda_r : H_r^{can}(M) \to H_{r-2}^{can}(M)$ is the homomorphism given by (4.26).

Next, we will obtain an explicit relation between the LJ-homology and the LJ-cohomology for the particular cases of a symplectic structure and of a Lie-Poisson structure.

4.4.1 Symplectic structures

Let (M, Ω) be a symplectic manifold of dimension $2m$. If f is a C^∞ real-valued function on M, it follows that $\mathcal{L}_{X_f}\Omega = 0$ which implies that

$$\mathcal{L}_{X_f}(\Omega \wedge \ldots^{(m} \ldots \wedge \Omega) = 0.$$

Thus, M is a unimodular Poisson manifold (see [55]). Using this fact, (3.19), Theorem 4.3 and Remark 4.4, we deduce the following result.

Theorem 4.11 *Let (M, Ω) be a symplectic manifold of dimension $2m$. Then*

$$H_k^{LJ}(M) \cong H_{LJ}^{2m-k+1}(M)$$

for all k. Moreover, if M is of type finite, we have

$$H_k^{LJ}(M) \cong \frac{H_{dR}^{2m-k+1}(M)}{\mathrm{Im} L^{2m-k-1}} \oplus \ker L^{2m-k},$$

where $H_{dR}^*(M)$ is the de Rham cohomology of M and $L^r : H_{dR}^r(M) \to H_{dR}^{r+2}(M)$ is the homomorphism given by (3.18).

4.4.2 Lie-Poisson structures

Let $(\mathfrak{g}, [\ ,\ ])$ be a real Lie algebra of dimension n and consider on the dual space $\mathfrak{g}^*$ the Lie-Poisson structure $\bar{\Lambda}$.

Suppose that $\{\xi_i\}_{i=1,\ldots,n}$ is a basis of $\mathfrak{g}$ and that (x_i) are the corresponding global coordinates for $\mathfrak{g}^*$. Denote by $\bar{\nu}$ the volume form on $\mathfrak{g}^*$ given by

$$\bar{\nu} = dx_1 \wedge \ldots \wedge dx_n$$

and by μ_0 the *modular character of* $\mathfrak{g}$, that is, μ_0 is the element of $\mathfrak{g}^*$ defined by

$$\mu_0(\xi) = \mathrm{trace}(ad_\xi), \qquad \text{for all } \xi \in \mathfrak{g},$$

where $ad_\xi : \mathfrak{g} \to \mathfrak{g}$ is the endomorphism given by

$$ad_\xi(\eta) = [\xi, \eta], \qquad \text{for all } \eta \in \mathfrak{g}.$$

μ_0 induces a constant vector field on $\mathfrak{g}^*$ which is the modular vector field of the Poisson manifold $(\mathfrak{g}^*, \bar{\Lambda})$ with respect to the volume form $\bar{\nu}$ (see [29, 55]). Thus, if $\mathfrak{g}$ is unimodular, i.e., if its modular character μ_0 is zero then $(\mathfrak{g}^*, \bar{\Lambda})$ is a unimodular Poisson manifold. Therefore, from Theorem 4.3 and Remark 4.4, we obtain

Theorem 4.12 *Let $(\mathfrak{g}, [\ ,\ ])$ be a unimodular real Lie algebra of dimension n and consider on the dual space $\mathfrak{g}^*$ the Lie-Poisson structure. Then, for all k,*

$$H_k^{LJ}(\mathfrak{g}^*) \cong H_{LJ}^{n-k+1}(\mathfrak{g}^*).$$

Now, using (3.29), Theorem 4.12 and the fact that the Lie algebra of a compact Lie group is unimodular, we conclude

Corollary 4.13 *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group of dimension n and consider on the dual space $\mathfrak{g}^*$ the Lie-Poisson structure. Then, for all k,*

$$H_k^{LJ}(\mathfrak{g}^*) \cong H_{LJ}^{n-k+1}(\mathfrak{g}^*) \cong (H^{n-k+1}(\mathfrak{g}) \otimes Inv) \oplus (H^{n-k}(\mathfrak{g}) \otimes Inv),$$

where Inv is the algebra of Casimir functions on $\mathfrak{g}^*$ and $H^*(\mathfrak{g})$ is the cohomology of $\mathfrak{g}$ relative to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$.

4.4.3 A quadratic Poisson structure

Let Λ be the quadratic Poisson structure on $\mathbb{R}^2$ considered in Section 3.3.3, that is,

$$\Lambda = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The modular vector field of $(\mathbb{R}^2, \Lambda)$ with respect to the standar volumen $\nu = dx \wedge dy$ is

$$\mathcal{X}_\Lambda^\nu = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

It is easy to confirm that $[\mathcal{X}_\Lambda^\nu] \neq 0$ in $H_{LP}^1(\mathbb{R}^2, \Lambda)$. Thus, $(\mathbb{R}^2, \Lambda)$ is not a unimodular Poisson manifold.

The canonical homology of $(\mathbb{R}^2, \Lambda)$. First, we will compute $H_2^{can}(\mathbb{R}^2, \Lambda)$.

Suppose that $\beta = hdx \wedge dy$ is a 2-form on $\mathbb{R}^2$, with $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$. We have that

$$\bar{\delta}\beta = -di(\Lambda)(\beta) = -d(xyh). \tag{4.27}$$

Hence, we deduce that

$$\bar{\delta}\beta = 0 \Leftrightarrow xyh = cte \Leftrightarrow h = 0 \Leftrightarrow \beta = 0.$$

Therefore,

$$H_2^{can}(\mathbb{R}^2, \Lambda) = \{0\}. \tag{4.28}$$

Now, let α be a 1-form on $\mathbb{R}^2$. It follows that

$$\bar{\delta}\alpha = i(\Lambda)(d\alpha) = xyd\alpha(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}).$$

Consequently,

$$\bar{\delta}\alpha = 0 \Leftrightarrow d\alpha = 0 \Leftrightarrow \alpha = df, \quad \text{with } f \in C^\infty(\mathbb{R}^2, \mathbb{R}).$$

This implies that (see (4.27))

$$H_1^{can}(\mathbb{R}^2, \Lambda) = \frac{\{df / f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}{\bar{\delta}(\Omega^2(\mathbb{R}^2))} = \frac{\{df / f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}{\{d(xyh) / h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}. \tag{4.29}$$

In particular, we obtain that the dimension of $H_1^{can}(\mathbb{R}^2, \Lambda)$ is not finite. In fact, if n is an arbitrary integer, $n \geq 1$, and

$$\sum_{k=1}^{n} \lambda_k [dx^k] = 0, \qquad \text{with } \lambda_k \in \mathbb{R},$$

we have that

$$\sum_{k=0}^{n} \lambda_k x^k = xyh,$$

where $\lambda_0 \in \mathbb{R}$ and $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Thus, we conclude that

$$\sum_{k=0}^{n} \lambda_k x^k = 0, \qquad \text{for all } x \in \mathbb{R},$$

and it follows that $\lambda_k = 0$, for all $k \in \{0, \ldots, n\}$ (note that $p(x) = \sum_{k=0}^{n} \lambda_k x^k$ is a polynomial of degree $\leq n$).

Finally, we will compute $H_0^{can}(\mathbb{R}^2, \Lambda)$.

If $\gamma = f dx + g dy$ is a 1-form on $\mathbb{R}^2$, we deduce that

$$\bar{\delta}\gamma = i(\Lambda)(d\gamma) = xy(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}),$$

which implies that

$$H_0^{can}(\mathbb{R}^2, \Lambda) = \frac{C^\infty(\mathbb{R}^2)}{\{xy(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) / f, g \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}.$$

On the other hand, using the fact that $H_{dR}^2(\mathbb{R}^2) = \{0\}$, we obtain that

$$\{xy(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) / f, g \in C^\infty(\mathbb{R}^2, \mathbb{R})\} = \{xyh / h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}$$

and therefore,

$$H_0^{can}(\mathbb{R}^2, \Lambda) = \frac{C^\infty(\mathbb{R}^2)}{\{xyh / h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}.$$

Now, we consider the $\mathbb{R}$-linear map

$$\psi : H_0^{can}(\mathbb{R}^2, \Lambda) \to H_1^{can}(\mathbb{R}^2, \Lambda) \oplus \mathbb{R}$$

defined by

$$\psi([f]) = ([df], f(0,0)), \qquad \text{for all } f \in C^\infty(\mathbb{R}^2, \mathbb{R}).$$

An straightforward computation shows that ψ is an isomorphism. In fact, the $\mathbb{R}$-linear map

$$\zeta : H_1^{can}(\mathbb{R}^2, \Lambda) \oplus \mathbb{R} \to H_0^{can}(\mathbb{R}^2, \Lambda)$$

given by

$$\zeta([df], k) = [f - f(0,0) + k]$$

is just the inverse of ψ.

Consequently,

$$H_0^{can}(\mathbb{R}^2, \Lambda) \cong \frac{\{df / f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}{\{d(xyh)/h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}} \oplus \mathbb{R}. \tag{4.30}$$

Remark 4.14 From (3.30), (4.28), (4.29) and (4.30), we conclude that

$$H_i^{can}(\mathbb{R}^2, \Lambda) \ncong H_{LP}^{2-i}(\mathbb{R}^2, \Lambda),$$

for $i \in \{0, 1, 2\}$.

The LJ-homology of $(\mathbb{R}^2, \Lambda)$. Using (4.28), (4.29), (4.30) and Theorem 4.9, we have that

$$\begin{aligned}
&H_3^{LJ}(\mathbb{R}^2, \Lambda, 0) \cong H_2^{can}(\mathbb{R}^2, \Lambda) = \{0\}, \\
&H_2^{LJ}(\mathbb{R}^2, \Lambda, 0) \cong H_1^{can}(\mathbb{R}^2, \Lambda) = \frac{\{df / f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}{\{d(xyh)/h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}, \\
&H_0^{LJ}(\mathbb{R}^2, \Lambda, 0) \cong H_0^{can}(\mathbb{R}^2, \Lambda) \cong \frac{\{df / f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}{\{d(xyh)/h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}} \oplus \mathbb{R}.
\end{aligned} \tag{4.31}$$

Next, we will show that

$$H_1^{LJ}(\mathbb{R}^2, \Lambda, 0) \cong H_1^{can}(\mathbb{R}^2, \Lambda) \oplus H_0^{can}(\mathbb{R}^2, \Lambda).$$

For this purpose, we consider the $\mathbb{R}$-linear map

$$\tilde{\psi} : H_1^{can}(\mathbb{R}^2, \Lambda) \oplus H_0^{can}(\mathbb{R}^2, \Lambda) \to H_1^{LJ}(\mathbb{R}^2, \Lambda, 0)$$

given by

$$\tilde{\psi}([\alpha], [f]) = [(\alpha, f)]$$

for $\alpha \in \Omega^1(\mathbb{R}^2)$ and $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$, with $\bar{\delta}\alpha = 0$.

Note that if

$$\alpha' = \alpha + \bar{\delta}\beta, \qquad f' = f + \bar{\delta}\gamma,$$

with $\beta \in \Omega^2(\mathbb{R}^2)$ and $\gamma \in \Omega^1(\mathbb{R}^2)$ then, since $H_{dR}^2(\mathbb{R}^2) = \{0\}$, there exists a 1-form $\tilde{\gamma}$ on $\mathbb{R}^2$ satisfying

$$\beta = -d\tilde{\gamma}$$

and

$$\delta(\beta, \gamma + \tilde{\gamma}) = (\bar{\delta}\beta, \bar{\delta}\gamma).$$

Thus,

$$(\alpha', f') = (\alpha, f) + \delta(\beta, \gamma + \tilde{\gamma}).$$

This proves that the map $\tilde{\psi}$ is well defined.

On the other hand, it is clear that $\tilde{\psi}$ is an epimorphism. Moreover, using again that $H^2_{dR}(\mathbb{R}^2) = \{0\}$, it follows that $\tilde{\psi}$ is a monomorphism.

Therefore (see (4.29) and (4.30))

$$H_1^{LJ}(\mathbb{R}^2, \Lambda, 0) \cong \left(\frac{\{df/f \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}{\{d(xyh)/h \in C^\infty(\mathbb{R}^2, \mathbb{R})\}}\right)^2 \oplus \mathbb{R}. \tag{4.32}$$

Remark 4.15 From (3.31), (4.31) and (4.32), we conclude that

$$H_i^{LJ}(\mathbb{R}^2, \Lambda, 0) \not\cong H_{LJ}^{3-i}(\mathbb{R}^2, \Lambda, 0), \qquad \text{for } i \in \{0, 1, 2, 3\}.$$

4.5 Lichnerowicz-Jacobi homology of a contact manifold

Let (M, η) be a contact manifold of dimension $2m+1$. Then $\nu = \eta \wedge (d\eta)^m$ is a volumen form and (see [53])

$$\mathcal{M}^\nu_{(\Lambda,E)} = (-(m+1)E, 0),$$

where (Λ, E) is the associated Jacobi structure on M. Thus, M is not a unimodular Jacobi manifold and therefore it is not possible to apply Theorem 4.3. In fact, in this section, we will show that the LJ-homology of M is trivial.

First, we prove the following result which will be useful in the sequel.

Lemma 4.16 *Let (M, η) be a contact manifold of dimension $2m+1$ and (Λ, E) be the associated Jacobi structure on M. If $e(\eta) : \Omega^k(M) \to \Omega^{k+1}(M)$ and $\tilde{L}^k : \Omega^k(M) \to \Omega^{k+2}(M)$ are the operators defined by*

$$e(\eta)(\alpha) = \eta \wedge \alpha, \qquad \tilde{L}^k(\alpha) = \alpha \wedge d\eta,$$

then,

$$i(\Lambda) \circ e(\eta) = e(\eta) \circ i(\Lambda), \tag{4.33}$$

$$i(\Lambda) \circ \tilde{L}^k - \tilde{L}^{k-2} \circ i(\Lambda) = (m-k)Id + e(\eta) \circ i_E, \tag{4.34}$$

where Id denotes the identity transformation.

Proof: Let x be an arbitrary point of M.

Suppose that $(t, q^1, \ldots, q^m, p_1, \ldots, p_m)$ are canonical coordinates in an open neighborhood U of x satisfying (2.9).

From (2.9), it follows that

$$(i(\Lambda) \circ e(\eta))(\alpha) = (e(\eta) \circ i(\Lambda))(\alpha),$$

for all $\alpha \in \Omega^k(U)$. This proves (4.33).

On the other hand, if $\alpha_1, \ldots, \alpha_k$ are 1-forms on U then a direct computation, using again (2.9), shows that

$$\begin{aligned}(i(\Lambda) \circ \tilde{L}^k - \tilde{L}^{k-2} \circ i(\Lambda))(\alpha_1 \wedge \ldots \wedge \alpha_k) &= m(\alpha_1 \wedge \ldots \wedge \alpha_k) - \sum_{i=1}^{k} \alpha_1 \wedge \ldots \wedge \alpha_{i-1} \\ &\wedge [\sum_{j=1}^{m} (\alpha_i(\frac{\partial}{\partial q^j} + p_j \frac{\partial}{\partial t}) dq^j + \alpha_i(\frac{\partial}{\partial p_j}) dp_j)] \wedge \alpha_{i+1} \wedge \ldots \wedge \alpha_k \\ &= (m-k)(\alpha_1 \wedge \ldots \wedge \alpha_k) + (e(\eta) \circ i_E)(\alpha_1 \wedge \ldots \wedge \alpha_k).\end{aligned}$$

Hence, we have (4.34). □

Now, we prove the following

Theorem 4.17 *The LJ-homology of a contact manifold is trivial.*

Proof: Let (M, η) be a contact manifold of dimension $2m + 1$ and δ the LJ-homology operator.

If α (respectively, β) is a k-form (respectively, a $(k-1)$-form) on M such that

$$\delta(\alpha, \beta) = (0, 0),$$

then, using (2.8), (4.5) and Lemma 4.16, we deduce that

$$(\alpha, \beta) = \delta(\frac{\eta \wedge \alpha}{m+1}, \frac{\eta \wedge \beta}{m+1}).$$

Therefore, $H_k^{LJ}(M) = \{0\}$. □

4.6 Lichnerowicz-Jacobi homology of a locally conformal symplectic manifold

As in Section 3.5, we will distinguish the two following cases:

i) The particular case of a g.c.s. manifold: Let (M, Ω) be a g.c.s. manifold with Lee 1-form ω. Then, there exists $f \in C^\infty(M, \mathbb{R})$ such that $\omega = df$ and the Jacobi structure of M is a conformal change of the Poisson structure on M associated with the symplectic form $e^{-f}\Omega$ (see Section 3.5). Thus, from Theorems 3.3, 4.7 and 4.11, we conclude

Theorem 4.18 *Let (M,Ω) be a g.c.s. manifold of dimension $2m$. Then,*

$$H_k^{LJ}(M) \cong H_{LJ}^{2m-k+1}(M)$$

for all k. Moreover, if M is of type finite, we have

$$H_k^{LJ}(M) \cong \frac{H_{dR}^{2m-k+1}(M)}{\mathrm{Im}\bar{L}^{2m-k-1}} \oplus \ker \bar{L}^{2m-k},$$

where $H_{dR}^*(M)$ is the de Rham cohomology of M and $\bar{L}^r : H_{dR}^r(M) \to H_{dR}^{r+2}(M)$ is the homomorphism given by

$$\bar{L}^r[\alpha] = [e^{-f}\alpha \wedge \Omega],$$

for all $[\alpha] \in H_{dR}^r(M)$.

Remark 4.19 In [53] Vaisman shows that a g.c.s. manifold is a unimodular Jacobi manifold. Using this fact, Theorem 3.11 and Theorem 4.3, we also can prove Theorem 4.18.

Example 4.20 Let (N,η) be a contact manifold of type finite. Assume that the dimension of N is $2m-1$ and consider on the product manifold $M = N \times \mathbb{R}$ the g.c.s. structure Ω given by (3.35) (see Example 3.13). Then, using Theorem 4.18, we have that

$$H_k^{LJ}(M) \cong H_{dR}^{2m-k+1}(M) \oplus H_{dR}^{2m-k}(M) \cong H_{dR}^{2m-k+1}(N) \oplus H_{dR}^{2m-k}(N).$$

ii) The general case: Now, we will study the LJ-homology of an arbitrary l.c.s. manifold.

Let (M,Ω) be a l.c.s. manifold of dimension $2m$ with Lee 1-form ω. Then $\nu = \Omega^m$ is a volumen form and (see [53])

$$\mathcal{M}^{\nu}_{(\Lambda,E)} = [(-(1+m)E, 0)],$$

where (Λ, E) the associated Jacobi structure on M. Thus, in general, M is not a unimodular Jacobi manifold (see [53]) and therefore it is not possible to apply Theorem 4.3. In fact, in this section, we will prove that if $k \in \{0,\ldots,2m+1\}$ then, in general, the spaces $H^k_{LJ}(M)$ and $H^{LJ}_{2m+1-k}(M)$, are not isomorphic.

First, we will introduce a certain cohomology in order to give an explicit description of the LJ-homology of M.

We consider the closed 1-forms ω_0 and ω_1 on M defined by

$$\omega_0 = -m\omega, \qquad \omega_1 = -(m+1)\omega. \tag{4.35}$$

Denote by $H^*_{\omega_0}(M)$ and $H^*_{\omega_1}(M)$ the cohomologies of the complexes $(\Omega^*(M), d_{\omega_0})$ and $(\Omega^*(M), d_{\omega_1})$, where d_{ω_0} and d_{ω_1} are the differential operators with zero square given by (see (3.36) and (3.37))

$$d_{\omega_0} = d + e(\omega_0), \qquad d_{\omega_1} = d + e(\omega_1). \tag{4.36}$$

Now, let $\tilde{d} : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \Omega^{k+1}(M) \oplus \Omega^k(M)$ be the differential operator defined by

$$\tilde{d}(\alpha, \beta) = (d_{\omega_1}\alpha - \Omega \wedge \beta, -d_{\omega_0}\beta). \tag{4.37}$$

Using (2.10), it follows $\tilde{d}^2 = 0$. Thus, we can consider the complex

$$\cdots \longrightarrow \Omega^{k-1}(M) \oplus \Omega^{k-2}(M) \xrightarrow{\tilde{d}} \Omega^k(M) \oplus \Omega^{k-1}(M) \xrightarrow{\tilde{d}} \Omega^{k+1}(M) \oplus \Omega^k(M) \longrightarrow \cdots$$

Denote by $\tilde{H}^*(M)$ the cohomology of this complex.

From (4.36) and (4.37), we deduce the following result which relates $\tilde{H}^*(M)$ with the cohomologies $H^*_{\omega_0}(M)$ and $H^*_{\omega_1}(M)$.

Proposition 4.21 *Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω. Suppose that $(Id^k, 0) : \Omega^k(M) \to \Omega^k(M) \oplus \Omega^{k-1}(M)$ and $(\pi_2)^k : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \Omega^{k-1}(M)$ are the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by*

$$(Id^k, 0)(\alpha) = (\alpha, 0), \qquad (\pi_2)^k(\alpha, \beta) = \beta,$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. Then:

(i) The mappings $(Id^k, 0)$ and $(\pi_2)^k$ induce an exact sequence of complexes

$$0 \longrightarrow (\Omega^*(M), d_{\omega_1}) \xrightarrow{(Id,0)} (\Omega^*(M) \oplus \Omega^{*-1}(M), \tilde{d}) \xrightarrow{\pi_2} (\Omega^{*-1}(M), -d_{\omega_0}) \longrightarrow 0.$$

(ii) This exact sequence induces a long exact cohomology sequence

$$\cdots \longrightarrow H^k_{\omega_1}(M) \xrightarrow{(Id^k,0)_*} \tilde{H}^k(M) \xrightarrow{((\pi_2)^k)_*} H^{k-1}_{\omega_0}(M) \xrightarrow{-L^{k-1}} H^{k+1}_{\omega_1}(M) \longrightarrow \cdots,$$

where the connector homomorphism $-L^{k-1}$ is defined by

$$(-L^{k-1})[\alpha] = [-\alpha \wedge \Omega], \tag{4.38}$$

for all $[\alpha] \in H^{k-1}_{\omega_0}(M)$.

Now, from Proposition 4.21, we obtain

Corollary 4.22 *Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω and such that the cohomology groups $H^k_{\omega_0}(M)$ and $H^k_{\omega_1}(M)$ have finite dimension, for all k. Then, the cohomology group $\tilde{H}^k(M)$ has also finite dimension, for all k, and*

$$\tilde{H}^k(M) \cong \frac{H^k_{\omega_1}(M)}{\mathrm{Im} L^{k-2}} \oplus \ker L^{k-1},$$

where $L^r : H^r_{\omega_0}(M) \to H^{r+2}_{\omega_1}(M)$ is the homomorphism given by (4.38).

Using (4.35), (4.36), Proposition 3.14, Theorem 3.15 and Corollary 4.22, we prove the following results:

Corollary 4.23 *Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω such that the dimensions of the cohomology groups $H^k_{\omega_0}(M)$ and $H^k_{\omega_1}(M)$ are finite, for all k. Suppose that Ω is $d_{(-\omega)}$-exact, that is, there exists a 1-form η on M satisfying*

$$\Omega = d\eta - \omega \wedge \eta.$$

Then, for all k, we have

$$\tilde{H}^k(M) \cong H^k_{\omega_1}(M) \oplus H^{k-1}_{\omega_0}(M).$$

Corollary 4.24 *Let (M, Ω) be a compact l.c.s. manifold with Lee 1-form $\omega \neq 0$. Suppose that g is a Riemannian metric on M such that ω is parallel with respect to g. Then, the cohomology $\tilde{H}^*(M)$ is trivial.*

Next, we will study the relation between the LJ-homology of a l.c.s. manifold M and the cohomology $\tilde{H}^*(M)$.

Let (M, Ω) be a l.c.s. manifold of dimension $2m$ with Lee 1-form ω. Denote by (Λ, E) the associated Jacobi structure on M and by $\#_\Lambda : \Omega^k(M) \to \mathcal{V}^k(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by (2.20) and (2.21).

We define the star operator $\star : \Omega^k(M) \to \Omega^{2m-k}(M)$ by

$$\star\, \alpha = (-1)^k i(\#_\Lambda(\alpha)) \frac{\Omega^m}{m!} \tag{4.39}$$

for all $\alpha \in \Omega^k(M)$.

Lemma 4.25 *If α is a k-form on M then:*

(i) $\star(\star(\alpha)) = \alpha$.

(ii) $(i_E \circ \star)(\alpha) = (-1)^k (\star \circ e(\omega))(\alpha)$.

(iii) $(\mathcal{L}_E \circ \star)(\alpha) = (\star \circ \mathcal{L}_E)(\alpha)$.

(iv) $(i(\Lambda) \circ \star)(\alpha) = \star(\alpha \wedge \Omega)$.

Proof: *(i)* Let $(q^1, \ldots, q^m, p_1, \ldots, p_m)$ be coordinates on an open subset U of M such that

$$\omega = df, \qquad \Omega = e^f \sum_i dq^i \wedge dp_i, \qquad \Lambda = e^{-f} \sum_i \left(\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}\right), \tag{4.40}$$

where $f : U \to \mathbb{R}$ is a C^∞ real-valued function on U (see Section 2.2).

Consider on U the symplectic 2-form

$$\bar{\Omega} = e^{-f}\Omega = \sum_i dq^i \wedge dp_i. \tag{4.41}$$

Denote by $\bar{\star} : \Omega^k(U) \to \Omega^{2m-k}(U)$ the star operator induced by $\bar{\Omega}$, that is,

$$\bar{\star}\alpha = (-1)^k i(\#_{\bar{\Lambda}}(\alpha))\frac{\bar{\Omega}^m}{m!},$$

where $\bar{\Lambda}$ is the Poisson structure associated with $\bar{\Omega}$.

In [5] (see also [35]), Brylinski shows that

$$\bar{\star}^2 = Id. \tag{4.42}$$

On the other hand, using (2.20), (2.21), (4.39), (4.40) and (4.41), we have that

$$\star\,(\alpha) = e^{(m-k)\sigma}\bar{\star}(\alpha), \qquad \text{for all } \alpha \in \Omega^k(U). \tag{4.43}$$

Thus, from (4.42) and (4.43), it follows that $\star^2(\alpha) = \alpha$, for all $\alpha \in \Omega^k(U)$. This proves (i).

Using (2.11), (3.52) and (4.39), we deduce (ii).

From (2.12), (4.39) and the first relation of (3.32), we deduce that (iii) holds.

Finally, (iv) follows directly using (4.39) and the fact that

$$\#_\Lambda(\Omega) = \Lambda.$$

□

The star operator $\bar{\star}$ induced by a symplectic form $\bar{\Omega}$ on a manifold M satisfies the following relation

$$\bar{\star}(\bar{\delta}\alpha) = (-1)^{k+1}d(\bar{\star}(\alpha)),$$

for all $\alpha \in \Omega^k(M)$, where $\bar{\delta} = i(\bar{\Lambda}) \circ d - d \circ i(\bar{\Lambda})$ is the canonical homology operator (see [5]). Therefore, proceeding as in the proof of the first part of Lemma 4.25 and using this lemma, we obtain

Lemma 4.26 *If α is a k-form on M then*

$$\star(i(\Lambda)d\alpha - di(\Lambda)\alpha) = (-1)^{k+1}(d(\star\alpha) - (m-k+1)e(\omega)(\star\alpha) - \Omega \wedge i_E(\star\alpha)).$$

Now, from Lemmas 4.25 and 4.26, we deduce the following

Theorem 4.27 *Let (M,Ω) be a l.c.s. manifold of dimension $2m$ and with Lee 1-form ω. Suppose that (Λ, E) is the associated Jacobi structure on M and that $\tilde{\phi}_k : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \Omega^{2m+1-k}(M) \oplus \Omega^{2m-k}(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$-modules defined by*

$$\tilde{\phi}_k(\alpha,\beta) = (\star\beta, i_E(\star\beta) - \star\alpha),$$

where $\star$ is the star operator given by (4.39). If δ is the LJ-homology operator of M and $\tilde{d}$ is the differential operator defined by (4.37) then,

$$\tilde{\phi}_{k-1}(\delta(\alpha,\beta)) = (-1)^k \tilde{d}(\tilde{\phi}_k(\alpha,\beta))$$

for all $(\alpha,\beta) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$. Thus,

$$H_k^{LJ}(M) \cong \tilde{H}^{2m+1-k}(M).$$

Using Theorem 4.27 and Corollaries 4.22, 4.23 and 4.24, we conclude

Corollary 4.28 *Let (M,Ω) be a l.c.s. manifold of dimension $2m$, with Lee 1-form ω and such that the cohomology groups $H^k_{\omega_0}(M)$ and $H^k_{\omega_1}(M)$ have finite dimension, for all k. Then:*

$$H_k^{LJ}(M) \cong \frac{H^{2m+1-k}_{\omega_1}(M)}{\mathrm{Im} L^{2m-k-1}} \oplus \ker L^{2m-k},$$

where $L^r : H^r_{\omega_0}(M) \to H^{r+2}_{\omega_1}(M)$ is the homomorphism given by (4.38).

Corollary 4.29 *Let (M,Ω) be a l.c.s. manifold of dimension $2m$, with Lee 1-form ω and such that the dimensions of the cohomology groups $H^k_{\omega_0}(M)$ and $H^k_{\omega_1}(M)$ are finite, for all k. Suppose that Ω is $d_{(-\omega)}$-exact, that is, there exits a 1-form η on M which satisfies*

$$\Omega = d\eta - \omega \wedge \eta.$$

Then,

$$H_k^{LJ}(M) \cong H^{2m+1-k}_{\omega_1}(M) \oplus H^{2m-k}_{\omega_0}(M),$$

for all k.

Corollary 4.30 *Let (M,Ω) be a compact l.c.s. manifold with Lee 1-form ω, $\omega \neq 0$. Suppose that g is a Riemannian metric on M such that ω is parallel with respect to g. Then, the LJ-homology of M is trivial.*

Remark 4.31 Note that under the same hypotheses as in Corollary 4.30, we have that $H^k_{LJ}(M) \cong H^k_{dR}(M)$, for all k.

Example 4.32 Let (N,η) be a compact contact manifold of dimension $2m-1$. We consider on the product manifold $M = N \times S^1$ the l.c.s. structure Ω given by (3.55) (see Example 3.21). Then, from Corollary 4.30, it follows that the LJ-homology of M is trivial.

4.7 Lichnerowicz-Jacobi homology of the unit sphere of a real Lie algebra

In this section we will give an explicit description of the LJ-homology of the unit sphere on the Lie algebra of a compact Lie group. For this purpose, we will prove that the unit sphere of a unimodular real Lie algebra is a unimodular Jacobi manifold.

Theorem 4.33 *Let $\mathfrak{g}$ be a unimodular real Lie algebra of dimension n. Suppose that $< , >$ is a scalar product on $\mathfrak{g}$ and consider on the unit sphere $S^{n-1}(\mathfrak{g})$ the induced Jacobi structure. Then $S^{n-1}(\mathfrak{g})$ is a unimodular Jacobi manifold. Thus,*

$$H_k^{LJ}(S^{n-1}(\mathfrak{g})) \cong H_{LJ}^{n-k}(S^{n-1}(\mathfrak{g})), \tag{4.44}$$

for all k.

Proof: Let $(x^i)_{i=1,\dots,n}$ be the global coordinates for $\mathfrak{g}$ obtained from an orthonormal basis $\{\xi_i\}_{i=1,\dots,n}$ of $\mathfrak{g}$.

Consider the volume form $\bar{\nu}$ on $\mathfrak{g}$ defined by

$$\bar{\nu} = dx^1 \wedge \dots \wedge dx^n.$$

Denote by $\flat_{< , >} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ the linear isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by (2.13) and by $\bar{\Lambda}$ the Poisson structure on $\mathfrak{g}$ induced by the Lie-Poisson structure on $\mathfrak{g}^*$ and by the isomorphism $\flat_{< , >}$.

Suppose that $(x_i)_{i=1,\dots,n}$ are the global coordinates for $\mathfrak{g}^*$ obtained from the basis $\{\xi_i\}_{i=1,\dots,n}$. Since $\flat^*_{< , >}(dx_1 \wedge \dots \wedge dx_n) = \bar{\nu}$, it follows that the modular vector field $\mathcal{X}^{\bar{\nu}}_{\bar{\Lambda}}$ of $(\mathfrak{g}, \bar{\Lambda})$ with respect to $\bar{\nu}$ is zero (see Section 4.4.2).

Now, consider the $(n-1)$-form ν on $S^{n-1}(\mathfrak{g})$ defined by

$$\nu = \sum_{i=1}^{n} (-1)^{n-i} < \xi_i, \ > d < \xi_1, \ > \wedge \dots \wedge d \widehat{< \xi_i, \ } > \wedge \dots \wedge d < \xi_n, \ >,$$

where $< \xi_j, \ >: S^{n-1}(\mathfrak{g}) \longrightarrow \mathbb{R}$ is the real function given by (2.17). Then, if $F : \mathfrak{g} - \{0\} \longrightarrow S^{n-1}(\mathfrak{g}) \times \mathbb{R}$ is the diffeomorphism defined by (2.19), a direct computation proves that

$$(F^{-1})^*(\bar{\nu}_{|\mathfrak{g}-\{0\}}) = e^{nt} \nu \wedge dt. \tag{4.45}$$

Therefore, ν is a volume form on $S^{n-1}(\mathfrak{g})$ and, using (4.45), Remarks 2.3 and 4.5 and the fact that the modular vector field of $(\mathfrak{g} - \{0\}, \bar{\Lambda}_{|\mathfrak{g}-\{0\}})$ with respect to $\bar{\nu}_{|\mathfrak{g}-\{0\}}$ is zero, we deduce that

$$\mathcal{M}^{\nu}_{(\Lambda, E)} = (0, 0),$$

(Λ, E) being the Jacobi structure of $S^{n-1}(\mathfrak{g})$. □

From Theorems 3.24 and 4.33, we conclude

Theorem 4.34 *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group of dimension n. Suppose that $< \, , \, >$ is a scalar product on $\mathfrak{g}$ and consider on the unit sphere $S^{n-1}(\mathfrak{g})$ the induced Jacobi structure. Then,*

$$H_k^{LJ}(S^{n-1}(\mathfrak{g})) \cong H_{LJ}^{n-k}(S^{n-1}(\mathfrak{g})) \cong H^{n-k}(\mathfrak{g}) \otimes Inv \tag{4.46}$$

for all k, where $H^*(\mathfrak{g})$ is the cohomology of $\mathfrak{g}$ relative to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$ and Inv is the subalgebra of $C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})$ defined by

$$Inv = \{\varphi \in C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R}) / X_f(\varphi) = 0, \ \ \forall f \in C^\infty(S^{n-1}(\mathfrak{g}), \mathbb{R})\}.$$

4.8 Table II

The following table summarizes the main results obtained in Sections 4.4, 4.5, 4.6 and 4.7 on the LJ-homology and its relation with the LJ-cohomology of the different types of Jacobi manifolds.

TYPE	LJ-HOMOLOGY	MODULAR CLASS	REMARKS
(M^{2m},Ω) symplectic of finite type	$H_k^{LJ}(M)\cong H_{LJ}^{2m+1-k}(M)$ $\cong \dfrac{H_{dR}^{2m+1-k}(M)}{\mathrm{Im}L^{2m-k-1}}\oplus\ker L^{2m-k}$	0	$L^r: H^r_{dR}(M)\to H^{r+2}_{dR}(M)$ $[\alpha]\mapsto[\alpha\wedge\Omega]$
Dual of a unimodular real Lie algebra $\mathfrak{g}$ of dimension n	$H_k^{LJ}(\mathfrak{g}^*)\cong H_{LJ}^{n-k+1}(\mathfrak{g}^*)$	0	
Dual of the Lie algebra $\mathfrak{g}$ of a compact Lie group $(\dim\mathfrak{g}=n)$	$H_k^{LJ}(\mathfrak{g}^*)\cong H_{LJ}^{n-k+1}(\mathfrak{g}^*)\cong$ $(H^{n-k+1}(\mathfrak{g})\otimes Inv)\oplus$ $\oplus(H^{n-k}(\mathfrak{g})\otimes Inv)$	0	$Inv\equiv$ subalgebra of the Casimir functions of $\mathfrak{g}^*$
M^{2m+1} contact	$H_k^{LJ}(M)=\{0\}$	$[-(m+1)E,0)]\neq 0$	$E\equiv$Reeb vector field
(M^{2m},Ω) g.c.s. of finite type with Lee 1-form $\omega=df$	$H_k^{LJ}(M)\cong$ $\cong \dfrac{H_{dR}^{2m+1-k}(M)}{\mathrm{Im}\bar{L}^{2m-k-1}}\oplus\ker\bar{L}^{2m-k}$	0	$\bar{L}^r: H^r_{dR}(M)\to H^{r+2}_{dR}(M)$ $[\alpha]\mapsto[e^{-f}\alpha\wedge\Omega]$
(M^{2m},Ω) l.c.s. with Lee 1-form ω and $dimH^*_{\omega_i}(M)<\infty$ $(i=0,1)$ $\omega_0=-m\omega,\ \ \omega_1=-(m+1)\omega$	$H_k^{LJ}(M)\cong$ $\cong \dfrac{H_{\omega_1}^{2m+1-k}(M)}{\mathrm{Im}L^{2m-k-1}}\oplus\ker L^{2m-k}$	$[(-(1+m)E,0)]$	$L^r: H^r_{\omega_0}(M)\to H^{r+2}_{\omega_1}(M)$ $[\alpha]\mapsto[\alpha\wedge\Omega]$
M^{2m} compact l.c.s. with Lee 1-form ω ω parallel with respect to a Riemannian metric	$H_k^{LJ}(M)=\{0\}$	$[(-(1+m)E,0)]\neq 0$	
Unit sphere $S^{n-1}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of a compact Lie group $(\dim\mathfrak{g}=n)$	$H_k^{LJ}(S^{n-1}(\mathfrak{g}))\cong$ $\cong H_{LJ}^{n-k}(S^{n-1}(\mathfrak{g}))\cong$ $H^{n-k}(\mathfrak{g})\otimes Inv$	0	$Inv\equiv$ subalgebra of the constant functions on the leaves of the characteristic foliation

Table II: LJ-homology

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